On two - dimensional complex Finsler manifolds

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Abstract

In this paper, we investigate the two-dimensional complex Finsler manifolds. The tools of this study are the complex Berwald frames $\{l, m, \bar{l}, \bar{m}\}$, $\{\lambda, \mu, \bar{\lambda}, \bar{\mu}\}$ and the Chern-Finsler connection with respect to these frames.

The geometry of two-dimensional complex Finsler manifolds is controlled by three real invariants which live on $T'M$: two horizontal curvature invariants $\mathbf{K}$ and $\mathbf{W}$ and one vertical curvature invariant $\mathbf{I}$. By means of these invariants are defined both the horizontal and the vertical holomorphic sectional curvatures in directions λ, μ and m, respectively.

The complex Landsberg and Berwald spaces are of particular interest. Complex Berwald spaces coincide with Kähler spaces, in the two - dimensional case. We establish the necessary and sufficient condition so that $\mathbf{K}$ is a constant and we obtain a characterization for the Kähler purely Hermitian spaces by the fact $\mathbf{K} = \mathbf{W} =$ constant and $\mathbf{I} = 0$. For the class of complex Berwald spaces we have $\mathbf{K} = \mathbf{W} = 0$. Finally, a classification of two - dimensional complex Finsler spaces for which the horizontal curvature satisfies a special property is obtained.

2000 Mathematics Subject Classification: 53B40, 53C60.
Key words and phrases: Berwald frame, complex Landsberg space, complex Berwald space, holomorphic sectional curvature.

1 Introduction

A great contribution to the geometry of two-dimensional real Finsler spaces is due to L. Berwald ([9], see also [12]). His theory is developed based on the choice of an orthonormal frame consisting of the normalized Liouville field and a unit field orthogonal to it. Many remarkable results are known for two-dimensional real Finsler spaces ([12, 6, 7, 8, 19]).

Part of the general themes from real Finsler geometry can be approached in complex Finsler geometry, the complex setting having the advantage of a powerful instrument, namely the Chern-Finsler connection (cf. [1]). This connection is Hermitian, of $(1, 0)$ - type and with other special properties, but as we expect, there are some uncomfortable computations on account of extending the theory to the complexified holomorphic tangent bundle $T'M$.

In a previous paper [17], we constructed the vertical Berwald frame in which the orthogonality is, with respect to the Hermitian structure, defined by the fundamental metric tensor of a 2 - dimensional complex Finsler space, on the holomorphic tangent manifold $T'M$.

The main purpose of this paper is to give a partial classification of the 2 - dimensional complex Finsler manifolds using its Chern-Finsler curvatures. We do not give a general complete classification, but we emphasize some important particular classes of the 2 - dimensional complex Finsler spaces.

Subsequently, we have made an overview of the paper's content.

In §2, we recall some preliminary properties of the n- dimensional complex Finsler spaces and complete with some others needed.

In §3, we prepare the tools for our aforementioned study. After we review from [17] the construction of the Berwald frame of a complex Finsler manifold of dimension two, we prefer to work in a fixed local chart in which there is obtained a local complex Berwald frame, which is extended to one on the horizontal part. We also find the expression of the complex Chern-Finsler connection with respect to these local frames. The independence of the obtained results from chosen chart is incessantly studied.

The local Berwald frames are not only a local geometrical machinery, but they also satisfy important properties which contain three main real invariants which live on $T'M$: one vertical curvature invariant $\mathbf{I}$ and two horizontal curvature invariants $\mathbf{K}$ and $\mathbf{W}$. By means of these invariants we are able to define and compute the horizontal and vertical holomorphic sectional curvatures in directions λ, μ and m, respectively. A first classification of the complex Finsler manifold of dimension two comes from the exploration of the $v\bar{v}-$, $h\bar{v}-$ and $v\bar{h}-$ Riemann type tensors, (Theorem 4.2). An immediate interest for the 2 - dimensional complex Berwald spaces is induced by the properties of the $h\bar{v}-$ and $v\bar{h}-$ Riemann type tensors. We prove that the two dimensional complex Berwald spaces are reduced to the Kähler spaces (Theorems 4.3, 4.4). We show that for the complex Berwald spaces $\mathbf{I}_{|k} = 0$, (Proposition 4.5). But this property is not peculiar only to the complex Berwald spaces. An example of the 2 - dimensional complex Finsler metric with $\mathbf{I}_{|k} = 0$, which is not Berwald, is given by the complex version of the Antonelli-Shimada metric. The necessary and sufficient conditions for 2 - dimensional complex Landsberg spaces are given in Theorem 4.6. Next, we

derive the Bianchi identities which specify the relations among the covariant derivatives of the three invariants and then use these relations to explore the holomorphic sectional curvatures. With some additional conditions of symmetry for the $h\bar{h}-$ Riemann type tensor, we find the necessary and sufficient conditions that $\mathbf{K}$ should be a constant (Theorems 4.8, 4.9). Moreover, we characterize the spaces with $\mathbf{K} = 0$ and $\mathbf{W} \leq 0$, (Theorem 4.11). The complex Berwald spaces with this symmetry have $\mathbf{K} = \mathbf{W} = 0$, (Theorem 4.12). It results that the Kähler purely Hermitian spaces are characterized by $\mathbf{K} = \mathbf{W} =$ constant and $\mathbf{I} = 0$, (Theorem 4.13). Finally, a special approach is devoted to the spaces with the $h\bar{h}-$ Riemann type tensor in the form $R_{\bar{r}j\bar{h}k} = \mathcal{K}(g_{j\bar{r}}g_{k\bar{h}} + g_{k\bar{r}}g_{j\bar{h}})$. We obtain two classes of such spaces, namely the Kähler purely Hermitian with $\mathcal{K}$ a constant and the complex spaces with $\mathcal{K} = 0$ and $\dot{\partial}_{\bar{h}}G^i = 0$, (Theorem 4.15). All these results are in §4.

2 Preliminaries

In the beginning, we will make a survey of complex Finsler geometry and we will set the basic notions and terminology. For more, see [1, 15].

Let M be a $n-$dimensional complex manifold, $z = (z^k)_{k=\overline{1,n}}$ are the complex coordinates in a local chart.

The complexified of the real tangent bundle $T_C M$ splits into the sum of holomorphic tangent bundle $T'M$ and its conjugate $T''M$. The bundle $T'M$ is itself a complex manifold, and the local coordinates in a local chart will be denoted by $u = (z^k, \eta^k)_{k=\overline{1,n}}$. They are changed into $(z'^k, \eta'^k)_{k=\overline{1,n}}$ by the rules $z'^k = z'^k(z)$ and $\eta'^k = \frac{\partial z'^k}{\partial z^l}\eta^l$.

A *complex Finsler space* is a pair (M, F), where $F : T'M \to \mathbb{R}^+$ is a continuous function satisfying the conditions:

i) $L := F^2$ is smooth on $\widetilde{T'M} := T'M \backslash \{0\}$;

ii) $F(z, \eta) \geq 0$, the equality holds if and only if $\eta = 0$;

iii) $F(z, \lambda\eta) = |\lambda| F(z, \eta)$ for $\forall \lambda \in \mathbb{C}$;

iv) the Hermitian matrix $\big(g_{i\bar{j}}(z, \eta)\big)$ is positive defined, where $g_{i\bar{j}} := \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$ is the fundamental metric tensor. Equivalently, it means that the indicatrix is strongly pseudo-convex.

Consequently, from *iii*) we have $\frac{\partial L}{\partial \eta^k}\eta^k = \frac{\partial L}{\partial \bar{\eta}^k}\bar{\eta}^k = L$, $\frac{\partial g_{i\bar{j}}}{\partial \eta^k}\eta^k = \frac{\partial g_{i\bar{j}}}{\partial \bar{\eta}^k}\bar{\eta}^k = 0$ and $L = g_{i\bar{j}}\eta^i\bar{\eta}^j$.

Roughly speaking, the geometry of a complex Finsler space consists of the study of the geometric objects of the complex manifold $T'M$ endowed with the Hermitian metric structure defined by $g_{i\bar{j}}$.

Therefore, the first step is to study the sections of the complexified tan-

gent bundle of $T'M$, which is decomposed in the sum $T_C(T'M) = T'(T'M) \oplus T''(T'M)$. Let $VT'M \subset T'(T'M)$ be the vertical bundle, locally spanned by $\{\frac{\partial}{\partial \eta^k}\}$, and $VT''M$ its conjugate.

At this point, the idea of complex nonlinear connection, briefly (*c.n.c.*), is an instrument in 'linearization' of this geometry. A (*c.n.c.*) is a supplementary complex subbundle to $VT'M$ in $T'(T'M)$, i.e. $T'(T'M) = HT'M \oplus VT'M$. The horizontal distribution $H_uT'M$ is locally spanned by $\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\}$, where $N_k^j(z,\eta)$ are the coefficients of the (*c.n.c.*). The pair $\{\delta_k := \frac{\delta}{\delta z^k}, \dot{\partial}_k := \frac{\partial}{\partial \eta^k}\}$ will be called the adapted frame of the (*c.n.c.*) which obey to the change rules $\delta_k = \frac{\partial z'^j}{\partial z^k}\delta'_j$ and $\dot{\partial}_k = \frac{\partial z'^j}{\partial z^k}\dot{\partial}'_j$. By conjugation, everywhere is obtained an adapted frame $\{\delta_{\bar{k}}, \dot{\partial}_{\bar{k}}\}$ on $T''_u(T'M)$. The dual adapted bases are $\{dz^k, \delta\eta^k\}$ and $\{d\bar{z}^k, \delta\bar{\eta}^k\}$.

Certainly, a main problem in this geometry is to determine a (*c.n.c.*) related only to the fundamental function of the complex Finsler space (M, F).

The next step is the action of a derivative law D on the sections of $T_C(T'M)$. First, let us consider the *Sasaki* type lift of the metric tensor $g_{i\bar{j}}$,

$$\mathcal{G} = g_{i\bar{j}} dz^i \otimes d\bar{z}^j + g_{i\bar{j}} \delta\eta^i \otimes \delta\bar{\eta}^j. \tag{2.1}$$

A Hermitian connection D, of $(1,0)-$ type, which satisfies in addition $D_{JX}Y = JD_XY$, for all X horizontal vectors and J the natural complex structure of the manifold, is the so called Chern-Finsler connection (cf. [1]), in brief $C - F$. The $C - F$ connection is locally given by the following coefficients (cf. [15]):

$$N_j^k = g^{\overline{m}k} \frac{\partial g_{l\overline{m}}}{\partial z^j} \eta^l = L_{lj}^k \eta^l \ ; \ L_{jk}^i = g^{\bar{l}i} \delta_k g_{j\bar{l}} \ \ ; \ C_{jk}^i = g^{\bar{l}i} \dot{\partial}_k g_{j\bar{l}} \ \ ; \ L_{\bar{j}k}^{\bar{\imath}} = C_{\bar{j}k}^{\bar{\imath}} = 0, \tag{2.2}$$

where here and further on δ_k is the adapted frame of the $C - F$ (*c.n.c.*) and $D_{\delta_k}\delta_j = L_{jk}^i \delta_i$, $D_{\dot{\partial}_k}\dot{\partial}_j = C_{jk}^i \dot{\partial}_i$, etc. The $C - F$ connection is the main tool in this study.

Denoting by " $_|$ " , " $|$ " , "$_{\bar{|}}$" and "$\bar{|}$", the $h-$, $v-$, $\overline{h}-$, $\overline{v}-$ covariant derivatives with respect to $C - F$ connection, respectively, for any X^i it results

$$\begin{aligned} X^i_{|k} \ &: = \delta_k X^i + X^l L^i_{lk} \ ; \ X^i|_k := \dot{\partial}_k \ X^i + X^l C^i_{lk}; \\ X^i_{|\bar{k}} \ &: = \delta_{\bar{k}} X^i \ ; \ X^i|_{\bar{k}} := \dot{\partial}_{\bar{k}} \ X^i; \end{aligned} \tag{2.3}$$

and

$$\eta^i_{|k} \ = \ \eta^i_{|\bar{k}} = \eta^i|_{\bar{k}} = 0 \ ; \ \ \eta^i|_k = \delta^i_k; \tag{2.4}$$

$$
\begin{aligned}
g_{i\bar{j}|k} &= g_{i\bar{j}|\overline{k}} = g_{i\bar{j}}|_k = g_{i\bar{j}}|_{\overline{k}} = 0;\\
(g_{i\bar{j}}\bar{\eta}^j)_{|k} &= (g_{i\bar{j}}\bar{\eta}^j)_{|\bar{k}} = (g_{i\bar{j}}\bar{\eta}^j)|_k = 0\ ;\ \ (g_{i\bar{j}}\bar{\eta}^j)|_{\bar{k}} = g_{i\bar{k}}.
\end{aligned}
$$

The nonzero curvatures of the $C-F$ connection are denoted by

$$
\begin{aligned}
R(\delta_h,\delta_{\bar{k}})\delta_j &= R^i_{j\bar{k}h}\delta_i\ ;\ R(\dot{\partial}_h,\delta_{\bar{k}})\delta_j = \Xi^i_{j\bar{k}h}\delta_i\ ;\ R(\delta_h,\dot{\partial}_{\bar{k}})\delta_j = P^i_{j\bar{k}h}\delta_i\\
R(\delta_h,\delta_{\bar{k}})\dot{\partial}_j &= R^i_{j\bar{k}h}\dot{\partial}_i\ ;\ R(\dot{\partial}_h,\delta_{\bar{k}})\dot{\partial}_j = \Xi^i_{j\bar{k}h}\dot{\partial}_i\ ;\ R(\delta_h,\dot{\partial}_{\bar{k}})\dot{\partial}_j = P^i_{j\bar{k}h}\dot{\partial}_i\\
R(\dot{\partial}_h,\dot{\partial}_{\bar{k}})\delta_j &= S^i_{j\bar{k}h}\delta_i\ ;\ \ R(\dot{\partial}_h,\dot{\partial}_{\bar{k}})\dot{\partial}_j = S^i_{j\bar{k}h}\dot{\partial}_i\ ,
\end{aligned}
$$

where

$$
\begin{aligned}
R^i_{j\overline{h}k} &= -\delta_{\overline{h}}L^i_{jk} - \delta_{\overline{h}}(N^l_k)C^i_{jl}\ \ ;\ \Xi^i_{j\overline{h}k} = -\delta_{\overline{h}}C^i_{jk} = \Xi^i_{k\overline{h}j};\\
P^i_{j\overline{h}k} &= -\dot{\partial}_{\overline{h}}L^i_{jk} - \dot{\partial}_{\overline{h}}(N^l_k)C^i_{jl}\ \ ;\ S^i_{j\overline{h}k} = -\dot{\partial}_{\overline{h}}C^i_{jk} = S^i_{k\overline{h}j}.
\end{aligned} \tag{2.5}
$$

Considering the Riemann tensor

$$
\begin{aligned}
\mathbf{R}(W,\overline{Z},X,\overline{Y}) &:= G(R(X,\overline{Y})W,\overline{Z}),\\
\mathbf{R}(W,\overline{Z},X,\overline{Y}) &= \overline{\mathbf{R}(Z,\overline{W},Y,\overline{X})}
\end{aligned} \tag{2.6}
$$

for W,X, $\overline{Z},\overline{Y}$ horizontal or vertical vectors, it results the $h\bar{h}-$, $h\bar{v}-$, $v\bar{h}-$, $v\bar{v}-$ Riemann type tensors: $R_{\bar{j}i\bar{h}k} = g_{l\bar{j}}R^l_{i\bar{h}k}$; $P_{\bar{j}i\bar{h}k} = g_{l\bar{j}}P^l_{i\bar{h}k}$; $\Xi_{\bar{j}i\bar{h}k} = g_{l\bar{j}}\Xi^l_{i\bar{h}k}$; $S_{\bar{j}i\bar{h}k} = g_{l\bar{j}}S^l_{i\bar{h}k}$, which have properties $R_{i\bar{j}k\overline{h}} = \overline{R_{\bar{j}i\bar{h}k}}$; $\Xi_{i\bar{j}k\overline{h}} = P_{\bar{j}i\bar{h}k}$; $P_{i\bar{j}k\overline{h}} = \Xi_{\bar{j}i\bar{h}k}$; $S_{i\bar{j}k\overline{h}} = S_{\bar{j}i\bar{h}k} = S_{\bar{h}i\bar{j}k}$, where $R_{i\bar{j}k\overline{h}} := \overline{R_{\bar{\imath}j\bar{k}h}}$, etc., (see [15], p. 77).

Further on, everywhere the index 0 means the contraction by η, for example $R^i_{0\overline{h}k} := R^i_{j\overline{h}k}\eta^j$.

Proposition 2.1. *i)* $R^i_{0\overline{h}k} = -\delta_{\overline{h}}N^i_k$; $\ R_{\bar{r}0\overline{h}k} = -g_{i\bar{r}}\delta_{\overline{h}}N^i_k$;

ii) $P^i_{0\overline{h}k} = -g^{\bar{m}i}C_{0\bar{m}\bar{h}|k}$; $\ P_{\bar{r}0\overline{h}k} = -C_{0\bar{r}\bar{h}|k}$; $P^i_{0\overline{0}k} = 0$;

iii) $\Xi^i_{j\overline{h}k} = -C^i_{jk|\bar{h}}$; $S^i_{j\overline{h}k} = -C^i_{jk}|_{\bar{h}}$; $\Xi^i_{0\overline{h}k} = \Xi^i_{k\overline{h}0} = S^i_{0\overline{h}k} = S^i_{k\overline{h}0}$;

$\Xi_{\bar{r}j\overline{h}k} = -C_{j\bar{r}k|\bar{h}}$; $S_{\bar{r}j\overline{h}k} = -C_{j\bar{r}k}|_{\bar{h}}$, *where we denoted* $C_{j\bar{r}k} := C^i_{jk}g_{i\bar{r}}$ *and* $C_{r\bar{j}\bar{k}}$ *is its conjugate*;

iv) $C_{l\bar{r}\bar{h}|k} = (\dot{\partial}_{\bar{h}}L^i_{lk})g_{i\bar{r}} + (\dot{\partial}_{\bar{h}}N^i_k)C_{i\bar{r}l}$;

v) $C_{l\bar{r}h|k} = (\dot{\partial}_h L^i_{lk})g_{i\bar{r}}$;

vi) $P^i_{j\overline{h}k} - P^i_{0\overline{h}k}|_j - P^i_{0\overline{h}r}C^r_{kj} = 0$.

Proof. i) and iii) results by (2.3), (2.4), (2.5) and $C^i_{0k} = C^i_{k0} = 0$.

For ii) we have

$$
\begin{aligned}
&P_{\bar{r}0\overline{h}k} = g_{i\bar{r}}P^i_{0\overline{h}k} = g_{i\bar{r}}\dot{\partial}_{\overline{h}}N^i_k = -g_{i\bar{r}}\dot{\partial}_{\bar{h}}\left(g^{\bar{m}i}\tfrac{\partial g_{j\bar{m}}}{\partial z^k}\eta^j\right)\\
&= g_{i\bar{r}}g^{\bar{m}l}g^{\bar{s}i}\left(\dot{\partial}_{\bar{h}}g_{l\bar{s}}\right)\tfrac{\partial g_{j\bar{m}}}{\partial z^k}\eta^j - g_{i\bar{r}}g^{\bar{m}i}\dot{\partial}_{\bar{h}}\left(\tfrac{\partial g_{j\bar{m}}}{\partial z^k}\right)\eta^j
\end{aligned}
$$

$= g^{\bar{m}l}\left(\dot{\partial}_{\bar{h}} g_{l\bar{r}}\right) \frac{\partial g_{j\bar{m}}}{\partial z^k}\eta^j - \frac{\partial}{\partial z^k}\left(\dot{\partial}_{\bar{h}} g_{j\bar{r}}\right)\eta^j = C_{l\bar{r}\bar{h}} N^l_k - \frac{\partial}{\partial z^k}\left(C_{j\bar{r}\bar{h}}\eta^j\right).$

Because $C_{0\overline{rh}} := C_{l\overline{rh}}\eta^l$ it leads to

$C_{0\overline{rh}|k} = (C_{l\overline{rh}}\eta^l)_{|k} = \delta_k(C_{l\overline{rh}}\eta^l) = \frac{\partial}{\partial z^k}\left(C_{l\overline{rh}}\eta^l\right) - N^s_k\dot{\partial}_s\left((\dot{\partial}_{\bar{h}} g_{l\bar{r}})\eta^l\right)$

$= \frac{\partial}{\partial z^k}\left(C_{l\overline{rh}}\eta^l\right) - N^s_k\dot{\partial}_{\bar{h}}\left((\dot{\partial}_s g_{l\bar{r}})\eta^l\right) - N^s_k C_{l\overline{rh}}\delta^l_s = \frac{\partial}{\partial z^k}\left(C_{l\overline{rh}}\eta^l\right) - N^s_k C_{s\overline{rh}}.$

From here, result the second relation of ii). The others immediately result by this.

Now, differentiating $N^i_k g_{i\bar{r}} = \frac{\partial g_{j\bar{r}}}{\partial z^k}\eta^j$ with respect to η^l yields $L^i_{lk} g_{i\bar{r}} = \frac{\partial g_{l\bar{r}}}{\partial z^k} - N^i_k C_{i\bar{r}l}$, which differentiated by $\bar{\eta}^h$ leads to iv).

Differentiating $L^i_{lk} g_{i\bar{r}} = \frac{\partial g_{l\bar{r}}}{\partial z^k} - N^i_k C_{i\bar{r}l}$, by $\bar{\eta}^h$ it results v).

It is obvious that $P^i_{0\bar{h}k} = -\dot{\partial}_{\bar{h}} N^i_k$. Hence,

$P^i_{j\bar{h}k} = -\dot{\partial}_{\bar{h}} L^i_{jk} - \dot{\partial}_{\bar{h}}(N^l_k) C^i_{jl} = -\dot{\partial}_{\bar{h}}(\dot{\partial}_j N^i_k) + P^l_{0\bar{h}k} C^i_{jl}$

$= -\dot{\partial}_j(\dot{\partial}_{\bar{h}} N^i_k) + P^l_{0\bar{h}k} C^i_{jl} = \dot{\partial}_j P^i_{0\bar{h}k} + P^l_{0\bar{h}k} C^i_{jl}$

$= P^i_{0\bar{h}k}|_j + P^i_{0\bar{h}r} C^r_{kj}$, i.e. vi). □

Proposition 2.2. *For any $X \in \Gamma^0(T'M)$ the following properties hold true:*

i) $X|_{k|j} - X_{|j}|_k = C^i_{jk} X_{|i}$;

ii) $X|_{\bar{k}|j} - X_{|j}|_{\bar{k}} = -P^i_{0\bar{k}j} X|_i$.

Proof. We have

$\left[\delta_j, \dot{\partial}_k\right] X = L^i_{kj}\left(\dot{\partial}_i X\right) = L^i_{kj} X|_i$ and

$\left[\delta_j, \dot{\partial}_{\bar{k}}\right] X = -P^i_{0\bar{k}j}\dot{\partial}_i X = -P^i_{0\bar{k}j} X|_i$.

On the other hand,

$\left[\delta_j, \dot{\partial}_k\right] X = \delta_j\left(\dot{\partial}_k X\right) - \dot{\partial}_k(\delta_j X) = \delta_j\left(X|_k\right) - \dot{\partial}_k(X_{|j})$

$= X|_{k|j} + L^i_{kj} X|_i - X_{|j}|_k - C^i_{jk} X_{|i}$ and

$\left[\delta_j, \dot{\partial}_{\bar{k}}\right] X = \delta_j\left(\dot{\partial}_{\bar{k}} X\right) - \dot{\partial}_{\bar{k}}(\delta_j X) = \delta_j\left(X|_{\bar{k}}\right) - \dot{\partial}_{\bar{k}}(X_{|j})$

$= X|_{\bar{k}|j} - X_{|j}|_{\bar{k}}$.

From the above relations it results i) and ii). □

For the vertical section $\mathcal{L} = \eta^k\dot{\partial}_k$, called the *Liouville* complex field (or the vertical radial vector field in [1]), we consider its horizontal lift $\chi := \eta^k\delta_k$.

According to [1], p. 108, [15], p. 81, the horizontal holomorphic curvature of the complex Finsler space (M, F) in direction η is

$$K_F(z,\eta) = \frac{2\mathcal{G}(\mathbf{R}(\chi,\bar{\chi})\chi,\bar{\chi})}{[\mathbf{G}(\chi,\bar{\chi})]^2} = \frac{2}{L^2}\mathcal{G}(\mathbf{R}(\chi,\bar{\chi})\chi,\bar{\chi}). \tag{2.7}$$

Let us recall that in [1]'s terminology, the complex Finsler space (M, F) is *strongly Kähler* iff $T^i_{jk} = 0$, *Kähler* iff $T^i_{jk}\eta^j = 0$ and *weakly Kähler* iff

$g_{i\bar{l}}T^i_{jk}\eta^j\overline{\eta}^l = 0$, where $T^i_{jk} := L^i_{jk} - L^i_{kj}$. In [10] it is proved that strongly Kähler and Kähler notions actually coincide. We notice that in the particular case of complex Finsler metrics which come from Hermitian metrics on M, so-called *purely Hermitian metrics* in [15], (i.e. $g_{i\bar{j}} = g_{i\bar{j}}(z)$), all those nuances of Kähler coincide.

It is well known by [1, 15] that the complex geodesics curves are defined by means of Chern-Finsler (*c.n.c.*). But this (*c.n.c.*) derives from a complex spray if the complex metric only is weakly Kähler. On the other hand, its local coefficients $N^k_j = g^{\bar{m}k}\frac{\partial g_{l\bar{m}}}{\partial z^j}\eta^l$, always determine a complex spray with coefficients $G^i = \frac{1}{2}N^i_j\eta^j$. Further on, G^i induce a (*c.n.c.*) by $\overset{c}{N^i_j} := \dot{\partial}_j G^i$ called *canonical* in [15], where it is proved that it coincides with Chern-Finsler (*c.n.c.*) if and only if the complex Finsler metric is Kähler. Using canonical (*c.n.c.*) we associate to it the next complex linear connections: one of Berwald type

$$B\Gamma := \left(\overset{c}{N^i_j}, \overset{B}{L^i_{jk}} := \dot{\partial}_k \overset{c}{N^i_j} = \overset{B}{L^i_{kj}}, \overset{B}{L^i_{j\bar{k}}} := \dot{\partial}_{\bar{k}} \overset{c}{N^i_j}, 0, 0\right)$$

and another of Rund type

$$R\Gamma := \left(\overset{c}{N^i_j}, \overset{c}{L^i_{jk}} := \frac{1}{2}g^{\bar{l}i}(\overset{c}{\delta_k} g_{j\bar{l}} + \overset{c}{\delta_j} g_{k\bar{l}}), \overset{c}{L^i_{j\bar{k}}} := \frac{1}{2}g^{\bar{l}i}(\overset{c}{\delta_{\bar{k}}} g_{j\bar{l}} - \overset{c}{\delta_{\bar{l}}} g_{j\bar{k}}), 0, 0\right),$$

where $\overset{c}{\delta_k} := \frac{\partial}{\partial z^k} - \overset{c}{N^j_k} \dot{\partial}_j$. $R\Gamma$ is only $h-$ metrical and $B\Gamma$ is neither $h-$ nor $v-$ metrical, (for more details see [15]). Note that $2G^i = N^i_j\eta^j = \overset{c}{N^i_j} \eta^j = \overset{B}{L^i_{jk}} \eta^j\eta^k$. Moreover, in the Kähler case we have $\overset{c}{\delta_k} = \delta_k$ and so, $L^i_{jk} = \overset{c}{L^i_{jk}} = \overset{B}{L^i_{jk}}$ and $\overset{c}{L^i_{j\bar{k}}} = 0$.

Further on, everywhere in this paper the Berwald and Rund connections will be specified by a super-index, like above (e.g. $\overset{c}{\delta_k}$, $\overset{B}{L^i_{jk}}$, $\overset{c}{L^i_{jk}}$, $X_{\overset{B}{|}k}$, etc.), while for the Chern-Finsler connection will be kept the initial generic notation without super-index (e.g. δ_k, L^i_{jk}, $X_{|k}$, etc.).

In the real case, a Finsler space is Landsberg if the Berwald and Rund connections coincide. Nevertheless, in complex Finsler geometry some differences appear. We speak about complex *Landsberg* space iff $\overset{B}{L^i_{jk}} = \overset{c}{L^i_{jk}}$, and about G - *Landsberg* space iff $\overset{B}{L^i_{jk}} = \overset{c}{L^i_{jk}}$ and the spray coefficients are holomorphic functions with respect to η, i.e. $\dot{\partial}_{\bar{k}}G^i = 0$, (see [5]).

Theorem 2.1. *([5]) Let (M, F) be a n - dimensional complex Finsler space. Then the following assertions are equivalent:*

i) (M, F) *is a* G *- Landsberg space;*

ii) $\overset{B}{L}{}^i_{jk} = \overset{c}{L}{}^i_{jk}(z)$;

iii) $C_{l\bar{r}h\overset{B}{|}0} = 0$ *and* $C_{r\bar{0}h\overset{B}{|}\bar{0}} = 0$, *where* $\overset{B}{|}$ *is* $h-$*covariant derivative with respect to* $B\Gamma$ *connection.*

We note that any complex Finsler space which is Kähler is Landsberg, too. So, by replacing the Landsberg condition from definition of the G - Landsberg space with the Kähler condition, we have obtained another class of complex Finsler spaces, called us G- *Kähler.* On the other hand, keeping with Aikou's work, a complex Finsler space which is Kähler and $L^i_{jk} = L^i_{jk}(z)$ is named complex *Berwald* space. Some tensorial characterizations for these classes of complex Finsler spaces are contained in the next theorem.

Theorem 2.2. *([5]) Let* (M, F) *be a* n *- dimensional complex Finsler space. Then the following assertions are equivalent:*

i) (M, F) *is* G *- Kähler;*

ii) $\overset{B}{L}{}^i_{j\bar{k}} = \overset{c}{L}{}^i_{j\bar{k}}$;

iii) (M, F) *is a complex Berwald space;*

iv) (M, F) *is a Kähler and either* $C_{l\bar{r}h|\bar{k}} = 0$ *or* $C_{l\bar{r}h|k} = 0$.

From Proposition 2.1 iii) and by $\Xi_{i\bar{j}k\bar{h}} = P_{\bar{j}i\bar{h}k}$, a complex Berwald space is a Kähler space with either $\Xi_{i\bar{j}k\bar{h}} = 0$ or $P_{\bar{j}i\bar{h}k} = 0$. Between above classes of complex Finsler spaces we have the inclusions: complex Berwald space $\subset$ G - Landsberg $\subset$ complex Landsberg space.

3 The complex Berwald frame

Let (M, F) be a 2 - dimensional complex Finsler space, $(z^k, \eta^k)_{k=\overline{1,2}}$ be complex coordinates on $T'M$ and $VT'M$ be the vertical bundle spanned by $\{\dot{\partial}_k\}$. Further on, the indices $i, j, k, ...$ run over $\{1, 2\}$. Let $g_{i\bar{j}}$ be the fundamental metric tensor of the space and $\mathcal{G}$ the Hermitian metric structure (2.1), defined on $T_C(T'M)$, with respect to the adapted frames of Chern-Finsler (*c.n.c.*).

We set $l := l^i\dot{\partial}_i$ and its dual form is $\omega = l_i\delta\eta^i$, where

$$l^i = \frac{1}{F}\eta^i \text{ and } l_i = \frac{1}{F}g_{i\bar{j}}\bar{\eta}^j = g_{i\bar{j}}l^{\bar{j}}. \tag{3.1}$$

Now, our aim is to construct an orthonormal frame in the vertical bundle $VT'M$, which is 2 - dimensional in any point. Therefore, it is decomposed into $VT'M = \{l\} \oplus \{l\}^\perp$, where $\{l\}^\perp$ is spanned by a complex vector m.

Requiring the orthogonality condition $\mathcal{G}(l, \bar{m}) = 0$ and $\mathcal{G}(m, \bar{m}) = 1$, i.e. m is a unit vector and, using $m_i := g_{i\bar{j}} m^{\bar{j}}$, the above two conditions get the linear system $\begin{cases} l_1 m^1 + l_2 m^2 = 0 \\ m_1 m^1 + m_2 m^2 = 1 \end{cases}$.

We try to solve this system following the same technique from [8] for real case. Nevertheless, let us pay more attention to this system. Passing in real coordinates, it contains three real equations with four real unknowns. So that it doesn't admit an unique solution. Formally, solving this system as one linear, it is obtained the 'solutions' $m^1 = \frac{-l_2}{\Delta}$, $m^2 = \frac{l_1}{\Delta}$, $m_1 = -\Delta l^2$ and $m_2 = \Delta l^1$, where $\Delta = l_1 m_2 - l_2 m_1$, which indeed are not completely determined because Δ depends on m_i. We can say more about these 'solutions'. A straightforward computation proves that $|\Delta| = \sqrt{g}$ and $\Delta' = \mathcal{T}\Delta$ under a change of the local coordinates $(z^k, \eta^k)_{k=\overline{1,2}}$ into $(z'^k, \eta'^k)_{k=\overline{1,2}}$, where $g := \det(g_{i\bar{j}})$ and $\mathcal{T} := \det\left(\frac{\partial z^i}{\partial z'^j}\right)$. Therefore, a natural question is if there exists at least Δ with the above mentioned properties. The answer will come below, when we find two distinct particular solutions for Δ.

Subsequently, our statement will be made for a fixed choice of Δ and then $\{l, m, \bar{l}, \bar{m}\}$ with

$$m = \frac{1}{\Delta}(-l_2 \dot{\partial}_1 + l_1 \dot{\partial}_2) \tag{3.2}$$

will be called the *complex Berwald frame*. Surely, the dependence of the chosen for Δ will be analyzed everywhere.

But when we work in a fixed local chart, we can choose $\Delta = \sqrt{g}$, i.e. Δ is real, which produces the unique solutions $m^1 = \frac{-l_2}{\sqrt{g}}$, $m^2 = \frac{l_1}{\sqrt{g}}$, $m_1 = -\sqrt{g} l^2$ and $m_2 = \sqrt{g} l^1$. Thus, we have

$$m = \frac{1}{\sqrt{g}}(-l_2 \dot{\partial}_1 + l_1 \dot{\partial}_2), \tag{3.3}$$

in this fixed chart.

Then $\{l, m, \bar{l}, \bar{m}\}$, with m given by (3.3) will be called the *local complex Berwald frame* of the space.

Note that (3.3) provides only a local frame, because the set of natural local basis in every chart does not have tensorial character. For this reason, considering a change of the local coordinates, we obtain

$$m' = \frac{\mathcal{T}}{|\mathcal{T}|} m \; ; \; m'^i = \frac{\mathcal{T}}{|\mathcal{T}|} \frac{\partial z'^i}{\partial z^k} m^k \; ; \; m'_i = \frac{\overline{\mathcal{T}}}{|\mathcal{T}|} \frac{\partial z^r}{\partial z'^i} m_r,$$

which show that m is not a vector, but it depends on the local change. Therefore, it will say that m from (3.3) is a pseudo-vector.

Although m from (3.3) depends on the local changes of the coordinates, it is very important in our study, in a fixed chart. Certainly, further on we will be very careful with the global validity of our assertions. We will see that together with its horizontal extension it gives rise to some invariants which will characterize two dimensional complex Finsler spaces. A first and useful remark is that the quantities $m_i m^j$, $m^i m^{\bar{j}}$, $m_i m_{\bar{j}}$ and $m_i m$ are independent of the chosen local chart, and hence they have global meaning.

With respect to the local complex Berwald frame, $\dot{\partial}_k$ and $g_{i\bar{j}}$ are decomposed as follows

$$\dot{\partial}_i = l_i l + m_i m \quad \text{and hence} \quad g_{i\bar{j}} = l_i l_{\bar{j}} + m_i m_{\bar{j}}. \tag{3.4}$$

From here we deduce that

$$C^i_{jk} = g^{\bar{m}i} \dot{\partial}_k g_{j\bar{m}} = A l^i m_k m_j + B m^i m_k m_j, \tag{3.5}$$

where we set

$$A := m^j m^k l_h C^h_{kj} \ ; \ B := m_h m^k m^j C^h_{jk}.$$

The dependence of the vertical terms A and B of the local charts is obvious, $A' = \frac{\mathcal{T}^2}{|\mathcal{T}|^2} A$; $B' = \frac{\mathcal{T}}{|\mathcal{T}|} B$. Thus, A and B are not invariants, but if they are zero in a local chart, then they are zero in any local chart. Moreover, by means of A and B and setting $\Delta = B\sqrt{g}$ with $|B|^2 = 1$ or $\Delta = \sqrt{Ag}$ with $|A|^2 = 1$, we obtain two particular solutions for m from (3.2) which certify the existence of the complex Berwald frames.

Further on, all our work will be with respect to the local complex Berwald frame, where m is given by (3.3).

Therefore, the formulas from Proposition 3.2, in [17], become

$$\begin{aligned}
l(l_i) &= \frac{-1}{2F} l_i \ ; \ \bar{l}(l_i) = \frac{1}{2F} l_i \ ; \ l(m_i) = \frac{1}{2F} m_i \ ; \ \bar{l}(m_i) = \frac{-1}{2F} m_i; \\
m(l_i) &= A m_i \ ; \ \bar{m}(l_i) = \frac{1}{F} m_i \ ; \ m(m_i) = \frac{1}{2} B m_i - \frac{1}{F} l_i; \ \bar{m}(m_i) = \frac{1}{2} \bar{B} m_i; \\
l(l^i) &= \frac{1}{2F} l^i \ ; \ \bar{l}(l^i) = -\frac{1}{2F} l^i \ ; \ l(m^i) = -\frac{1}{2F} m^i \ ; \ \bar{l}(m^i) = \frac{1}{2F} m^i; \\
m(l^i) &= \frac{1}{F} m^i \ ; \ \bar{m}(l^i) = 0 \ ; \\
m(m^i) &= -\frac{1}{2} B m^i - A l^i \ ; \ \bar{m}(m^i) = -\frac{1}{F} l^i - \frac{1}{2} \bar{B} m^i.
\end{aligned} \tag{3.6}$$

By a direct computation, using the above relations, we obtain formulas for the vertical covariant derivatives of $l, m, \bar{l}$ and $\bar{m}$ with respect to the $C-F$

connection

$$
\begin{aligned}
l_i|_j &= \frac{-1}{2F}l_il_j + Am_im_j;\ \ l_i|_{\bar{j}} = \frac{1}{2F}l_il_{\bar{j}} + \frac{1}{F}m_im_{\bar{j}}; \qquad (3.7)\\
m_i|_j &= \frac{1}{2F}m_il_j - \frac{1}{F}l_im_j - \frac{B}{2}m_im_j\ ;\ \ m_i|_{\bar{j}} = \frac{-1}{2F}m_il_{\bar{j}} + \frac{\bar{B}}{2}m_im_{\bar{j}};\\
l^i|_j &= \frac{1}{F}\delta^i_j - \frac{1}{2F}l_jl^i\ ;\ \ l^i|_{\bar{j}} = \frac{-1}{2F}l_{\bar{j}}l^i\ ;\ F|_j = \frac{1}{2}l_j;\\
m^i|_j &= \frac{-1}{2F}l_jm^i + \frac{B}{2}m_jm^i\ ;\ \ m^i|_{\bar{j}} = \frac{1}{2F}l_{\bar{j}}m^i - \frac{1}{F}m_{\bar{j}}l^i - \frac{\bar{B}}{2}m_{\bar{j}}m^i,
\end{aligned}
$$

and their conjugates.

Moreover, because $\bar{l}(C^h_{kj}) = 0$ and $l(C^h_{kj}) = -\frac{1}{F}C^h_{kj}$ by some computation, it results

$$
\begin{aligned}
A|_{\bar{h}} &= \dot{\partial}_{\bar{h}}A = (l_{\bar{h}}\bar{l} + m_{\bar{h}}\bar{m})A = \frac{3A}{2F}l_{\bar{h}} + A|_{\bar{s}}m^{\bar{s}}m_{\bar{h}}; \qquad (3.8)\\
B|_{\bar{h}} &= \dot{\partial}_{\bar{h}}B = (l_{\bar{h}}\bar{l} + m_{\bar{h}}\bar{m})B = \frac{B}{2F}l_{\bar{h}} + B|_{\bar{s}}m^{\bar{s}}m_{\bar{h}};\\
A|_h &= \dot{\partial}_hA = (l_hl + m_hm)\,A = -\frac{5A}{2F}l_h + A|_sm^sm_h;\\
B|_h &= \dot{\partial}_hB = (l_hl + m_hm)\,B = -\frac{3B}{2F}l_h + B|_sm^sm_h.
\end{aligned}
$$

Now, via the natural isomorphism between the bundles $VT'M$ and $T'M$, composed with the horizontal lift of $HT'M$, we obtain the following orthonormal local frame on $H_CT'M$,

$$
\{\lambda := l^i\delta_i,\ \ \mu = m^i\delta_i,\ \ \bar{\lambda} := l^{\bar{i}}\delta_{\bar{i}},\ \ \bar{\mu} = m^{\bar{i}}\delta_{\bar{i}}\}.
$$

Let D be the $C-F$ connection on (M,F). Further on, let us give an explicit expression for $C-F$ connection with respect to horizontal local frame $\{\lambda,\mu,\bar{\lambda},\bar{\mu}\}$. Moreover, using (3.4) and $L^i_{jk} = g^{\bar{m}i}\delta_kg_{j\bar{m}}$ it results

$$
\begin{aligned}
L^i_{jk} &= Jl^il_jl_k + Ul^im_jl_k + Vl^il_jm_k + Xl^im_jm_k \qquad (3.9)\\
&\quad + Om^il_jl_k + Ym^im_jl_k + Em^il_jm_k + Hm^im_jm_k,
\end{aligned}
$$

where we set

$$
\begin{aligned}
J\ &:= l^jl^kl_iL^i_{jk};\ U := m^jl^kl_iL^i_{jk};\ V := l^jm^kl_iL^i_{jk};\ X := m^jm^kl_iL^i_{jk} \qquad (3.10)\\
O\ &:= l^jl^km_iL^i_{jk};\ Y := m^jl^km_iL^i_{jk};\ E := l^jm^km_iL^i_{jk};\ H := m^jm^km_iL^i_{jk}.
\end{aligned}
$$

Here the horizontal settled quantities do not have tensorial character, because under the change of charts we have

$$\begin{aligned} J' &= J + \mathcal{T}^r_{ab} l^a l^b l_r \ ; \ U' = \frac{\mathcal{T}}{|\mathcal{T}|}(U + \mathcal{T}^r_{ab} m^a l^b l_r) \ ; \\ V' &= \frac{\mathcal{T}}{|\mathcal{T}|}(V + \mathcal{T}^r_{ab} l^a m^b l_r) \ ; \ X' = \frac{\mathcal{T}^2}{|\mathcal{T}|^2}(X + \mathcal{T}^r_{ab} m^a m^b l_r) \ ; \\ O' &= \frac{\overline{\mathcal{T}}}{|\mathcal{T}|}(O + \mathcal{T}^r_{ab} l^a l^b m_r) \ ; \ Y' = Y + \mathcal{T}^r_{ab} m^a l^b m_r \ ; \\ E' &= E + \mathcal{T}^r_{ab} l^a m^b m_r \ ; \ H' = \frac{\mathcal{T}}{|\mathcal{T}|}(H + \mathcal{T}^r_{ab} m^a m^b m_r), \end{aligned} \tag{3.11}$$

where $\mathcal{T}^r_{ab} := \frac{\partial z'^j}{\partial z^a}\frac{\partial z'^k}{\partial z^b}\frac{\partial^2 z^r}{\partial z'^j \partial z'^k}$.

Firstly, the properties of the $C-F$ connection $N^i_k = L^i_{jk}\eta^j$ and $\dot{\partial}_j N^i_k = L^i_{jk}$, (see [15]), permit us to establish some links between the vertical and horizontal terms (3.10) of this connection. Indeed,

$N^i_k = F(Jl^i l_k + Vl^i m_k + Om^i l_k + Em^i m_k)$ and

$L^i_{jk} = (l_j l + m_j m)[F(Jl^i l_k + Vl^i m_k + Om^i l_k + Em^i m_k)]$

$= [\frac{1}{2}J + Fl(J)]l^i l_j l_k + [Fm(J) - V - FAO]l^i m_j l_k + [Fl(V) + \frac{3}{2}V]l^i l_j m_k$

$+[Fm(V) + FAJ + \frac{1}{2}FBV - FAE]l^i m_j m_k + [Fl(O) - \frac{1}{2}O]m^i l_j l_k$

$+[Fm(O) + J - \frac{1}{2}FBO - E]m^i m_j l_k + [Fl(E) + \frac{1}{2}E]m^i l_j m_k$

$+[Fm(E) + V + FAO]m^i m_j m_k$ which together with (3.9) give,

Proposition 3.1. *Let (M, F) be a 2 - dimensional complex Finsler space. Then*

i) $J|_k = \frac{1}{2F}Jl_k + [\frac{1}{F}(U+V) + AO]m_k$;

ii) $V|_k = -\frac{1}{2F}Vl_k + [A(E-J) - \frac{1}{2}BV + \frac{1}{F}X]m_k$;

iii) $O|_k = \frac{3}{2F}Ol_k + [\frac{1}{F}(E+Y-J) + \frac{1}{2}BO]m_k$;

iv) $E|_k = \frac{1}{2F}El_k + [\frac{1}{F}(H-V) - AO]m_k$.

Proof. In the fixed local chart the assertions i)-iv) are true. We must prove their global validity. For example, under the change of a local chart, we have

$V'|'_k + \frac{1}{2F}V'l'_k - [A'(E'-J') - \frac{1}{2}B'V' + \frac{1}{F}X']m'_k$

$= \frac{\mathcal{T}}{|\mathcal{T}|}\frac{\partial z^r}{\partial z'^k}\{V|_r + \frac{1}{2F}Vl_r - [A(E-J) - \frac{1}{2}BV + \frac{1}{F}X]m_r\}$, where $V'|'_k := \dot{\partial}'_k V'$.

Because $V|_r + \frac{1}{2F}Vl_r - [A(E-J) - \frac{1}{2}BV + \frac{1}{F}X]m_r = 0$, by its change rule it results that it is zero in any local chart. Analogous results the geometric character of the others assertions. □

Proposition 3.2. *Let (M, F) be a 2 - dimensional complex Finsler space. Then*

i) It is Kähler if and only if $U = V$ and $Y = E$;

ii) It is weakly Kähler if and only if $U = V$.

Proof. i) By (3.9), $L^i_{jk} - L^i_{kj} = (U-V)l^i m_j l_k + (V-U)l^i l_j m_k + (Y-E)m^i m_j l_k + (E-Y)m^i l_j m_k$. So, $L^i_{jk} - L^i_{kj} = 0$ if and only if $U = V$ and $Y = E$.

To prove ii) we compute $g_{i\bar{l}}T^i_{jk}\eta^j\overline{\eta}^l = F^2(L^i_{jk} - L^i_{kj})l_i l^j = F^2(V-U)m_k$. It results $g_{i\bar{l}}T^i_{jk}\eta^j\overline{\eta}^l = 0$ if and only if $U = V$.

Taking into account the local changes of $U - V$ and $Y - E$, it follows the global validity of these statements. □

Further on, several calculus imply the following properties.

Proposition 3.3. *With respect to the local Berwald frame, we have:*

$$\begin{aligned}
\lambda(l_i) &= Jl_i + Um_i \; ; \; \bar{\lambda}(l_i) = \bar{\lambda}(l^i) = 0 \; ; \; \lambda(l^i) = -Jl^i - Om^i \; ; \qquad (3.12)\\
\lambda(m_i) &= Ol_i - \frac{1}{2}(J-Y)m_i \; ; \; \bar{\lambda}(m_i) = \frac{1}{2}(\bar{J}+\bar{Y})m_i \; ;\\
\lambda(m^i) &= -Ul^i + \frac{1}{2}(J-Y)m^i \; ; \; \bar{\lambda}(m^i) = -\frac{1}{2}(\bar{J}+\bar{Y})m^i \; ;\\
\mu(l_i) &= Vl_i + Xm_i \; ; \; \bar{\mu}(l_i) = \bar{\mu}(l^i) = 0 \; ; \; \mu(l^i) = -Vl^i - Em^i \; ;\\
\mu(m_i) &= El_i + \frac{1}{2}(H-V)m_i \; ; \; \bar{\mu}(m_i) = \frac{1}{2}(\bar{V}+\bar{H})m_i \; ;\\
\mu(m^i) &= -Xl^i - \frac{1}{2}(H-V)m^i \; ; \; \bar{\mu}(m^i) = -\frac{1}{2}(\bar{V}+\bar{H})m^i \; ;\\
\lambda(g) &= (J+Y)g \; ; \; \mu(g) = (V+H)g \; ; \; \delta_i = l_i\lambda + m_i\mu \; ; \; \lambda(L) = \mu(L) = 0
\end{aligned}$$

and their conjugates.

Then, from (3.12) we deduce that

$$\begin{aligned}
l_{i|j} &= l_{i|\bar{j}} = l^i_{|j} = l^i_{|\bar{j}} = 0; \qquad (3.13)\\
m_{i|j} &= -\frac{1}{2}[(J+Y)l_j + (V+H)m_j]m_i; \; m_{i|\bar{j}} = \frac{1}{2}[(\bar{J}+\bar{Y})l_{\bar{j}} + (\bar{V}+\bar{H})m_{\bar{j}}]m_i;\\
m^i_{|j} &= \frac{1}{2}[(J+Y)l_j + (V+H)m_j]m^i; \; m^i_{|\bar{j}} = -\frac{1}{2}[(\bar{J}+\bar{Y})l_{\bar{j}} + (\bar{V}+\bar{H})m_{\bar{j}}]m^i
\end{aligned}$$

and theirs conjugates.

4 Curvatures of the C-F connection

In this section, we shall compute the curvature coefficients of the $C - F$ connection with respect to the local frames $\{l, m, \bar{l}, \bar{m}\}$ and $\{\lambda, \mu, \bar{\lambda}, \bar{\mu}\}$. By means of these, we characterize the 2 - dimensional complex Finsler spaces.

4.1 The $v\bar{v}-$ Riemann type tensor

Firstly, we study the $v\bar{v}-$ Riemann type tensor $S_{\bar{r}j\bar{h}k}$. Taking into account Proposition 2.1 iii) and the formulas (3.5), (3.7) and (3.8), we have

$S_{\bar{r}j\bar{h}k} = -(Al_{\bar{r}}m_jm_k + Bm_{\bar{r}}m_jm_k)|_{\bar{h}}$
$= [-A|_{\bar{h}} + \frac{3A}{2F}l_{\bar{h}} + (-A\bar{B} + \frac{B}{F})m_{\bar{h}}]l_{\bar{r}}m_jm_k$
$+(-B|_{\bar{h}} + \frac{B}{2F}l_{\bar{h}} - \frac{B\bar{B}}{2}m_{\bar{h}})m_{\bar{r}}m_jm_k$
$= (-A|_{\bar{s}}m^{\bar{s}} - A\bar{B} + \frac{B}{F})m_{\bar{h}}l_{\bar{r}}m_jm_k + (-B|_{\bar{s}}m^{\bar{s}} - \frac{B\bar{B}}{2})m_{\bar{h}}m_{\bar{r}}m_jm_k.$

But, $S_{\bar{r}j\bar{h}k}$ is symmetric in j, k and $\bar{r}, \bar{h}$. Therefore, it results that

$$S_{\bar{r}j\bar{h}k} = \mathbf{I}m_{\bar{h}}m_{\bar{r}}m_jm_k\ ;\ \ A|_{\bar{s}}m^{\bar{s}} = -A\bar{B} + \frac{B}{F}, \tag{4.1}$$
$$\text{where}\quad \mathbf{I} := \quad -B|_{\bar{s}}m^{\bar{s}} - \frac{B\bar{B}}{2}.$$

We note that $\mathbf{I}$ is invariable to the changes of the local coordinates thanks to $S_{\bar{r}j\bar{h}k}$ and $m_jm_{\bar{h}}m_km_{\bar{r}}$ which are tensors. Further on, we point out some properties of the function $\mathbf{I}$, called by us the *vertical curvature invariant.*

By analogy with (2.7), we define *the vertical holomorphic sectional curvature* in direction l

$$K^v_{F,l}(z,\eta) := 2\mathbf{R}(l,\bar{l},l,\bar{l}) \tag{4.2}$$

and *the vertical holomorphic sectional curvature* in direction m

$$K^v_{F,m}(z,\eta) := 2\mathbf{R}(m,\bar{m},m,\bar{m}) \tag{4.3}$$

Theorem 4.1. *Let (M,F) be a 2 - dimensional complex Finsler space. Then*
i) $K^v_{F,l}(z,\eta) = 0$;
ii) $K^v_{F,m}(z,\eta) = 2\mathbf{I}$ and $\mathbf{I}$ is real valued;
iii) $\mathbf{I}|_0 = -\mathbf{I}$.

Proof. By (4.1) $\mathbf{R}(l,\bar{l},l,\bar{l}) = l^{\bar{h}}l^{\bar{r}}l^jl^kS_{\bar{r}j\bar{h}k} = 0$ and $\mathbf{R}(m,\bar{m},m,\bar{m}) = m^{\bar{h}}m^{\bar{r}}m^jm^kS_{\bar{r}j\bar{h}k} = \mathbf{I}$. Indeed, $\bar{\mathbf{I}} = \overline{\mathbf{R}(m,\bar{m},m,\bar{m})} = \mathbf{R}(m,\bar{m},m,\bar{m}) = \mathbf{I}$. These imply i) and ii).

Considering the Bianchi identity $S_{\bar{r}j\bar{h}k}|_i = S_{\bar{r}j\bar{h}i}|_k$, (see [15], p. 77) and using the relations (3.5) and (4.1), we have

$\mathbf{I}|_im_{\bar{h}}m_{\bar{r}}m_jm_k - \frac{1}{F}\mathbf{I}m_{\bar{h}}m_{\bar{r}}m_jm_il_k = \mathbf{I}|_km_{\bar{h}}m_{\bar{r}}m_jm_i - \frac{1}{F}\mathbf{I}m_{\bar{h}}m_{\bar{r}}m_jm_kl_i$, which contracted by $m^{\bar{h}}m^{\bar{r}}m^jm^il^k$ gives iii). □

Proposition 4.1. *Let (M,F) be a 2 - dimensional complex Finsler space.*
i) It is purely Hermitian if and only if $A = 0$;
ii) If $|A| \neq 0$ and $B = 0$ then $\mathbf{I} = 0$ and $A|_{\bar{h}} = \frac{3A}{2F}l_{\bar{h}}$.

Proof. By (4.1), $A = 0$ implies $B = 0$. These give $C_{i\bar{h}j} = 0$ which means that $\frac{\partial g_{i\bar{h}}}{\partial \eta^j} = 0$, i.e. F is purely Hermitian. Conversely, if F is purely Hermitian then $A = B = 0$. Thus, the assertion i) is proved.

The claim ii) follows readily from (4.1) and (3.8). Obviously, the statements are independent of the changes of local charts. □

The above Proposition shows that there are 2 - dimensional complex Finsler spaces with $K^v_{F,m}(z, \eta) = 0$ which are not purely Hermitian. Subsequently, we pay more attention to the case $AB^2 \neq 0$.

4.2 The $v\bar{h}-$ Riemann type tensor

Let $\Xi_{\bar{r}j\bar{h}k}$ be the $v\bar{h}-$ Riemann type tensor. Using the Proposition 2.1 iii) and the formulas (3.5) and (3.13), we have

$$\begin{aligned}\Xi_{\bar{r}j\bar{h}k} &= -[A_{|\bar{h}}l_{\bar{r}} + A(\bar{J} + \bar{Y})l_{\bar{r}}l_{\bar{h}} + A(\bar{V} + \bar{H})l_{\bar{r}}m_{\bar{h}} \\ &+B_{|\bar{h}}m_{\bar{r}} + \frac{B}{2}(\bar{J} + \bar{Y})m_{\bar{r}}l_{\bar{h}} + \frac{B}{2}(\bar{V} + \bar{H})m_{\bar{r}}m_{\bar{h}}]m_j m_k.\end{aligned} \tag{4.4}$$

We wish to investigate the relationship among A, B, $\mathbf{I}$ and to characterize the 2 - dimensional complex Finsler spaces by means of these. For this, contracting the Bianchi identity

$$\Xi_{\bar{r}j\bar{h}k}|_{\bar{s}} - S_{\bar{r}j\bar{s}k|\bar{h}} + \Xi_{\bar{r}j\bar{p}k}\overline{C^p_{sh}} = 0, \tag{4.5}$$

(see [15], p. 77), with the tensor $m^{\bar{r}}m^j m^k m^{\bar{s}}$ and taking into account (4.4) and (3.7), we obtain

$\Xi_{\bar{r}j\bar{h}k}|_{\bar{s}}m^{\bar{r}}m^j m^k m^{\bar{s}} = -\{B_{|\bar{h}}|_{\bar{s}}m^{\bar{s}} + \frac{\bar{B}}{2}B_{|\bar{h}}$
$+\frac{1}{2}[-\mathbf{I}(\bar{J} + \bar{Y}) + B(\bar{J} + \bar{Y})|_{\bar{s}}m^{\bar{s}} - \frac{B}{F}(\bar{V} + \bar{H})]l_{\bar{h}}$
$+\frac{1}{2}[B|_{\bar{s}}m^{\bar{s}}(\bar{V} + \bar{H}) + B(\bar{V} + \bar{H})|_{\bar{s}}m^{\bar{s}}]m_{\bar{h}}\};$
$S_{\bar{r}j\bar{s}k|\bar{h}}m^{\bar{r}}m^j m^{\bar{s}}m^k = \mathbf{I}_{|\bar{h}};$
$\Xi_{\bar{r}j\bar{p}k}\overline{C^p_{sh}}m^{\bar{r}}m^j m^k m^{\bar{s}} = -[\frac{\bar{A}}{F}B_{|\bar{0}} + \bar{B}B_{|\bar{p}}m^{\bar{p}} + \frac{\bar{A}B}{2}(\bar{J} + \bar{Y}) + \frac{B\bar{B}}{2}(\bar{V} + \bar{H})]m_{\bar{h}}.$
Hence

$$\begin{aligned}B_{|\bar{h}}|_{\bar{s}}m^{\bar{s}} &= -\{\frac{1}{2}[-\mathbf{I}(\bar{J} + \bar{Y}) + B(\bar{J} + \bar{Y})|_{\bar{s}}m^{\bar{s}} - \frac{B}{F}(\bar{V} + \bar{H})]l_{\bar{h}} \\ &+\frac{1}{2}[(-\mathbf{I}+\frac{B\bar{B}}{2})(\bar{V} + \bar{H}) + B(\bar{V} + \bar{H})|_{\bar{s}}m^{\bar{s}} \\ &+2\frac{\bar{A}}{F}B_{|\bar{0}} + 2\bar{B}B_{|\bar{p}}m^{\bar{p}} + \bar{A}B(\bar{J} + \bar{Y})]m_{\bar{h}} + \mathbf{I}_{|\bar{h}} + \frac{\bar{B}}{2}B_{|\bar{h}}\}\end{aligned} \tag{4.6}$$

and its conjugate.

On the other hand, contracting in (4.5) by $m^{\bar r}m^j m^k l^{\bar s}$, using
$\Xi_{\bar r j\bar h k}|_{\bar s} m^{\bar r}m^j m^k l^{\bar s} = -\frac{1}{F}\{B_{|\bar h}|_{\bar 0} - \frac{1}{2}B_{|\bar h} + \frac{B}{2}[-\frac{1}{2}(\bar J+\bar Y) + (J+Y)|_{\bar 0}]l_{\bar h}$
$+\frac{B}{2}[\frac{1}{2}(\bar V+\bar H) + (\bar V+\bar H)|_{\bar 0}]m_{\bar h}\}$ and
$S_{\bar r j\bar s k|\bar h} m^{\bar r}m^j l^{\bar s} m^k = \Xi_{\bar r j\bar p k}\overline{C^p_{sh}} m^{\bar r}m^j m^k l^{\bar s} = 0$
we have,

$$\begin{aligned} B_{|\bar h}|_{\bar 0} \;=\; & \frac{1}{2}B_{|\bar h} - \frac{B}{2}[-\frac{\bar J+\bar Y}{2} + (\bar J+\bar Y)|_{\bar 0}]l_{\bar h} \\ & -\frac{B}{2}[\frac{\bar V+\bar H}{2} + (\bar V+\bar H)|_{\bar 0}]m_{\bar h}, \end{aligned} \tag{4.7}$$

and its conjugate.

The conjugates of (4.6), (4.7) and Theorem 4.1 ii) allow us to write

$$\begin{aligned} \bar B_{|k}|_j \;=\; & \frac{1}{2F}\{\bar B_{|k} - \bar B[-\frac{J+Y}{2} + (J+Y)|_0]l_k \\ & -\bar B[\frac{V+H}{2} + (V+H)|_0]m_k\}l_j \\ & -\{\frac{1}{2}[-\mathbf{I}(J+Y) + \bar B(J+Y)|_s m^s - \frac{\bar B}{F}(V+H)]l_k \\ & +\frac{1}{2}[(-\mathbf{I}+\frac{B\bar B}{2})(V+H) + \bar B(V+H)|_s m^s \\ & +2\frac{A}{F}\bar B_{|0} + 2B\bar B_{|s}m^s + A\bar B(J+Y)]m_k + \mathbf{I}_{|k} + \frac{B}{2}\bar B_{|k}\}m_j. \end{aligned} \tag{4.8}$$

It is also worthwhile to note the following identity

$$\begin{aligned} \bar B|_{j|k} \;=\; & \frac{1}{2F}\bar B_{|k}l_j \\ & -\{\mathbf{I}_{|k} + \frac{\bar B}{2}B_{|k} + \frac{B}{2}\bar B_{|k} + \frac{1}{2}(-\mathbf{I}-\frac{B\bar B}{2})[(J+Y)l_k + (V+H)m_k]\}m_j, \end{aligned} \tag{4.9}$$

which is obtained from (3.7), (3.8) and (4.1).

Therefore, (4.8) and (4.9) lead to

$$\begin{aligned} \bar B|_{j|k} - \bar B_{|k}|_j \;=\; & C^i_{jk}\bar B_{|i} + \frac{\bar B}{2}\{\frac{1}{F}[-\frac{J+Y}{2} + (J+Y)|_0]l_k l_j \\ & +\frac{1}{F}[\frac{V+H}{2} + (V+H)|_0]m_k l_j \\ & +[\frac{B}{2}(J+Y) + (J+Y)|_s m^s - \frac{1}{F}(V+H)]l_k m_j \\ & +[B(V+H) + (V+H)|_s m^s + A(J+Y)]m_k m_j - B_{|k}m_j\}, \end{aligned} \tag{4.10}$$

because $C^i_{jk}\bar B_{|i} = (\frac{A}{F}\bar B_{|0} + B\bar B_{|s}m^s)m_k m_j$.

4.3 The $h\bar{v}-$ Riemann type tensor

Now let us consider the $h\bar{v}-$ Riemann type tensor $P_{\bar{r}j\bar{h}k}$. By Proposition 2.1.ii) and formulas (3.5) and (3.7), it results that

$$P_{\bar{r}0\bar{h}k} = -F[\bar{A}_{|k} + \bar{A}(J+Y)l_k + \bar{A}(V+H)m_k]m_{\bar{r}}m_{\bar{h}}. \tag{4.11}$$

But, Proposition 2.1 vi) allows us to reconstruct $P_{\bar{r}j\bar{h}k}$. Indeed,

$$P_{\bar{r}j\bar{h}k} = P_{\bar{r}0\bar{h}k}|_j + P_{\bar{r}0\bar{h}s}C^s_{kj} \tag{4.12}$$

and from (4.11), we obtain

$$\begin{aligned} P_{\bar{r}0\bar{h}s}C^s_{kj} \;=\; & -F[\frac{A}{F}\bar{A}_{|0} + B\bar{A}_{|s}m^s \\ & +A\bar{A}(J+Y) + B\bar{A}(V+H)]m_{\bar{r}}m_{\bar{h}}m_k m_j \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} P_{\bar{r}0\bar{h}k}|_j \;=\; & -\{-\frac{1}{2}\bar{A}_{|k}l_j + FB\bar{A}_{|k}m_j + F\bar{A}_{|k}|_j - \bar{A}(J+Y)l_k l_j \\ & +\bar{A}[FB(J+Y) - (V+H)]l_k m_j + \frac{F\bar{A}B}{2}(V+H)m_k m_j \\ & +F[\bar{A}|_j(J+Y) + \bar{A}(J+Y)|_j]l_k \\ & +F[\bar{A}|_j(V+H) + \bar{A}(V+H)|_j]m_k\}m_{\bar{r}}m_{\bar{h}}. \end{aligned} \tag{4.14}$$

Plugging (4.13) and (4.14) into (4.12), gives

$$\begin{aligned} P_{\bar{r}j\bar{h}k} \;=\; & -\{-\frac{1}{2}\bar{A}_{|k}l_j + FB\bar{A}_{|k}m_j + F\bar{A}_{|k}|_j - \bar{A}(J+Y)l_k l_j \\ & +\bar{A}[FB(J+Y) - (V+H)]l_k m_j \\ & +[A\bar{A}_{|0} + FB\bar{A}_{|s}m^s + FA\bar{A}(J+Y) + \frac{3F\bar{A}B}{2}(V+H)]m_k m_j \\ & +F[\bar{A}|_j(J+Y) + \bar{A}(J+Y)|_j]l_k \\ & +F[\bar{A}|_j(V+H) + \bar{A}(V+H)|_j]m_k\}m_{\bar{r}}m_{\bar{h}}. \end{aligned} \tag{4.15}$$

Recall the following property, $P_{\overline{\bar{r}j\bar{h}k}} = \Xi_{j\bar{r}k\bar{h}} = \overline{\Xi_{\bar{j}r\bar{k}h}}$. Writing it by means of (4.4) and (4.15), we obtain the conditions

$$\overline{A}_{|k}|_0 \;=\; \frac{3}{2}\overline{A}_{|k}\;; \tag{4.16}$$

$$\begin{aligned} \overline{A}_{|k}|_j m^j \;=\; & \frac{1}{F}\overline{B}_{|k} - B\overline{A}_{|k} - [\frac{\bar{B}}{2F}(J+Y) + \bar{A}(J+Y)|_s m^s - \frac{\bar{A}}{F}(V+H)]l_k \\ & -[(\frac{\bar{B}}{2F} + \frac{\bar{A}B}{2})(V+H) + \bar{A}(V+H)|_s m^s + A\bar{A}(J+Y) \\ & +\frac{A}{F}\overline{A}_{|0} + B\overline{A}_{|s}m^s]m_k \end{aligned}$$

and theirs conjugates.

From both formulas (4.16), it follows that

$$\begin{aligned}\overline{A}_{|k}|_j &= \frac{3}{2F}\overline{A}_{|k}l_j + \{\frac{1}{F}\overline{B}_{|k} - B\overline{A}_{|k} \\ &-[\frac{\bar{B}}{2F}(J+Y) + \bar{A}(J+Y)|_s m^s - \frac{\bar{A}}{F}(V+H)]l_k \\ &-[(\frac{\bar{B}}{2F} + \frac{\bar{A}B}{2})(V+H) + \bar{A}(V+H)|_s m^s + A\bar{A}(J+Y) \\ &+\frac{A}{F}\overline{A}_{|0} + B\overline{A}_{|s}m^s]m_k\}m_j.\end{aligned} \tag{4.17}$$

Moreover, from (3.7), (3.8) and (4.1), we have

$$\begin{aligned}\overline{A}|_{j|k} &= \frac{3}{2F}\overline{A}_{|k}l_j + \{-B\overline{A}_{|k} - \overline{A}B_{|k} + \frac{1}{F}\overline{B}_{|k} \\ &-(\frac{\bar{B}}{2F} - \frac{\bar{A}B}{2})[(J+Y)l_k + (V+H)m_k]\}m_j.\end{aligned} \tag{4.18}$$

By subtracting (4.17) from (4.18), we get

$$\begin{aligned}\overline{A}|_{j|k} - \overline{A}_{|k}|_j &= C^i_{jk}\overline{A}_{|i} \\ &-\overline{A}\{B_{|k} - [\frac{B}{2}(J+Y) + (J+Y)|_s m^s - \frac{1}{F}(V+H)]l_k \\ &-[B(V+H) + (V+H)|_s m^s + A(J+Y)]m_k\}m_j,\end{aligned} \tag{4.19}$$

because $C^i_{jk}\overline{A}_{|i} = (\frac{A}{F}\overline{A}_{|0} + B\overline{A}_{|s}m^s)m_k m_j$.

Theorem 4.2. *Let (M,F) be a 2 - dimensional complex Finsler space. Then it is*

i) purely Hermitian, or

ii) with $|A| \neq 0$, $B = 0$ and

$$(J+Y)|_s m^s = \frac{1}{F}(V+H)\ ; \ \ (V+H)|_s m^s = -A(J+Y), \tag{4.20}$$

or

iii) with $AB^2 \neq 0$ and

$$\begin{aligned}(J+Y)|_0 &= \frac{J+Y}{2}\ ; \ \ (V+H)|_0 = -\frac{V+H}{2}\ ; \\ B_{|k} &= [\frac{B}{2}(J+Y) + (J+Y)|_s m^s - \frac{1}{F}(V+H)]l_k \\ &+[B(V+H) + (V+H)|_s m^s + A(J+Y)]m_k.\end{aligned} \tag{4.21}$$

Proof. Writing the identity i) from Proposition 2.2 for the vertical terms $\overline{A}$ and $\overline{B}$ it involves $\overline{A}|_{k|j} - \overline{A}_{|j}|_k = C^i_{jk}\overline{A}_{|i}$ and $\overline{B}|_{k|j} - \overline{B}_{|j}|_k = C^i_{jk}\overline{B}_{|i}$. But, taking into account (4.19) and (4.10), it follows

$$\begin{aligned}&\overline{A}\{B_{|k} - [\frac{B}{2}(J+Y) + (J+Y)|_s m^s - \frac{1}{F}(V+H)]l_k \\ &-[B(V+H) + (V+H)|_s m^s + A(J+Y)]m_k\}m_j = 0\end{aligned} \tag{4.22}$$

and

$$\begin{aligned}&\bar{B}\{\frac{1}{F}[-\frac{J+Y}{2} + (J+Y)|_0]l_k l_j \\ &+\frac{1}{F}[\frac{V+H}{2} + (V+H)|_0]m_k l_j \\ &+[\frac{B}{2}(J+Y) + (J+Y)|_s m^s - \frac{1}{F}(V+H)]l_k m_j \\ &+[B(V+H) + (V+H)|_s m^s + A(J+Y)]m_k m_j - B_{|k} m_j\} = 0.\end{aligned} \tag{4.23}$$

Hence, we have the cases:
1. If $\overline{A} = 0$ then by means of Proposition 4.1 i) gives the statement i), or
2. If $\overline{A} \neq 0$ then $|A| \neq 0$ and by (4.22) we obtain

$$\begin{aligned}B_{|k} \;=\; &[\frac{B}{2}(J+Y) + (J+Y)|_s m^s - \frac{1}{F}(V+H)]l_k \\ &+[B(V+H) + (V+H)|_s m^s + A(J+Y)]m_k,\end{aligned} \tag{4.24}$$

which substituted into (4.23) leads to

$$\bar{B}\{[-\frac{J+Y}{2} + (J+Y)|_0]l_k l_j + [\frac{V+H}{2} + (V+H)|_0]m_k l_j\} = 0. \tag{4.25}$$

From here, it results either $\bar{B} = 0$ which together with (4.24) gives ii) or $\bar{B} \neq 0$. In this last case we have $|B| \neq 0$, and by (4.24) results $(J+Y)|_0 = \frac{J+Y}{2}$ and $(V+H)|_0 = -\frac{V+H}{2}$, which with (4.23) imply iii).

The independence of the above statement to the changes of local charts results by straightforward computations using (3.11). □

4.4 Two dimensional complex Berwald and Landsberg spaces

The above considerations offer us the premises for some special characterizations of the 2 - dimensional complex Berwald and Landsberg spaces. Firstly, we write the identity iv) of Proposition 2.1 in terms of the local complex Berwald frame. Some computations give

$\dot{\partial}_{\bar{h}} L^i_{jk} = \{[\bar{l}(J) + \frac{1}{2F}J]l^i l_j l_k + [\bar{l}(U) - \frac{1}{2F}U]l^i m_j l_k + [\bar{l}(V) - \frac{1}{2F}V]l^i l_j m_k$
$+[\bar{l}(X) - \frac{1}{2F}X]l^i m_j m_k + [\bar{l}(O) + \frac{3}{2F}O]m^i l_j l_k + [\bar{l}(Y) + \frac{1}{2F}Y]m^i m_j l_k$
$+[\bar{l}(E) + \frac{1}{2F}E]m^i l_j m_k + [\bar{l}(H) - \frac{1}{2F}H]m^i m_j m_k\}l_{\bar{h}}$
$+\{[\bar{m}(J) - \frac{1}{F}O]l^i l_j l_k + [\bar{m}(U) - \frac{1}{F}(Y - J) + \frac{1}{2}\bar{B}U]l^i m_j l_k$
$+[\bar{m}(V) - \frac{1}{F}(E - J) + \frac{1}{2}\bar{B}V]l^i l_j m_k + [\bar{m}(X) - \frac{1}{F}(H - U - V) + \bar{B}X]l^i m_j m_k$
$+[\bar{m}(O) - \frac{1}{2}\bar{B}O]m^i l_j l_k + [\bar{m}(Y) + \frac{1}{F}O]m^i m_j l_k + [\bar{m}(E) + \frac{1}{F}O]m^i l_j m_k$
$+[\bar{m}(H) + \frac{1}{F}(Y + E) + \frac{1}{2}\bar{B}H]m^i m_j m_k\}m_{\bar{h}}.$

Using $\Xi_{\bar{r}j\bar{h}k} = -C_{j\bar{r}k|\bar{h}}$ and (4.4) it results

$C_{j\bar{r}k|\bar{h}} = [A_{|\bar{h}}l_{\bar{r}} + A(\bar{J}+\bar{Y})l_{\bar{r}}l_{\bar{h}} + A(\bar{V}+\bar{H})l_{\bar{r}}m_{\bar{h}} + B_{|\bar{h}}m_{\bar{r}} + \frac{B}{2}(\bar{J}+\bar{Y})m_{\bar{r}}l_{\bar{h}} + \frac{B}{2}(\bar{V}+\bar{H})m_{\bar{r}}m_{\bar{h}}]m_j m_k.$

The above outcomes substituted into Proposition 2.1 iv), lead to

Proposition 4.2. *Let (M, F) be a 2 - dimensional complex Finsler space. Then*

i) $J|_{\bar{k}} = -\frac{1}{2F}Jl_{\bar{k}} + \frac{1}{F}Om_{\bar{k}}$; $V|_{\bar{k}} = \frac{1}{2F}Vl_{\bar{k}} + [\frac{1}{F}(E - J) - \frac{1}{2}\bar{B}V]m_{\bar{k}}$;
ii) $\bar{l}(U) - \frac{1}{2F}U = \bar{l}(X) - \frac{1}{2F}X = \bar{l}(O) + \frac{3}{2F}O = \bar{l}(Y) + \frac{1}{2F}Y = \bar{l}(E) + \frac{1}{2F}E$
$= \bar{l}(H) - \frac{1}{2F}H = 0$;
iii) $\bar{m}(U) - \frac{1}{F}(Y - J) + \frac{1}{2}\bar{B}U + FA[\bar{m}(O) - \frac{1}{2}\bar{B}O] = 0$;
iv) $\bar{m}(V) - \frac{1}{F}(E - J) + \frac{1}{2}\bar{B}V = 0$;
v) $\bar{m}(X) - \frac{1}{F}(H - U - V) + \bar{B}X + FA[\bar{m}(E) + \frac{1}{F}O] = 0$;
vi) $\frac{1}{F}\bar{A}_{|0} + \bar{A}(J + Y) = \bar{m}(O) - \frac{1}{2}\bar{B}O$;
vii) $\frac{1}{F}\bar{B}_{|0} + \frac{\bar{B}}{2}(J + Y) = \bar{m}(Y) + \frac{1}{F}O + FB[\bar{m}(O) - \frac{1}{2}\bar{B}O]$;
viii) $\bar{A}_{|k}m^k + \bar{A}(V + H) = \bar{m}(E) + \frac{1}{F}O$;
ix) $\bar{B}_{|k}m^k + \frac{\bar{B}}{2}(V + H) = \bar{m}(H) + \frac{1}{F}(Y + E) + \frac{1}{2}\bar{B}H + FB[\bar{m}(E) + \frac{1}{F}O]$.

Next, we rewrite the identity v) from Proposition 2.1, $\dot{\partial}_h L^i_{jk} = C_{j\bar{r}h|k}g^{\bar{r}i}$ with respect to the complex Berwald frame. Taking into account Proposition 3.1, we have

$\dot{\partial}_h L^i_{jk} = \{[l(U) + \frac{1}{2F}U]l^i m_j l m_k + [l(X) + \frac{3}{2F}X]l^i m_j m_k$
$+[l(Y) - \frac{1}{2F}Y]m^i m_j l_k + [l(H) + \frac{1}{2F}H]m^i m_j m_k\}l_h$
$+\{[m(U) - A(Y - J) + \frac{1}{2}BU - \frac{1}{F}X]l^i m_j l_k + [m(Y) + AO - \frac{1}{F}(H - U)]m^i m_j l_k$
$+[m(X) + A(U + V - H) + BX]l^i m_j m_k$
$+[m(H) + A(Y + E) + \frac{1}{F}X + \frac{1}{2}BH]m^i m_j m_k\}m_h.$

On the other hand, $C_{j\bar{r}h|k}g^{\bar{r}i} = \{[A_{|k} - A(J + Y)l_k - A(V + H)m_k]l^i$
$+[B_{|k} - \frac{B}{2}(J + Y)l_k - \frac{B}{2}(V + H)m_k]m^i\}m_j m_h$. From here we obtain

Proposition 4.3. *Let (M, F) be a 2 - dimensional complex Finsler space. Then*

i) $l(U) + \frac{1}{2F}U = l(X) + \frac{3}{2F}X = l(Y) - \frac{1}{2F}Y = l(H) + \frac{1}{2F}H = 0$;
ii) $m(U) - A(Y - J) + \frac{1}{2}BU - \frac{1}{F}X = \frac{1}{F}A_{|0} - A(J + Y)$;
iii) $m(Y) + AO - \frac{1}{F}(H - U) = \frac{1}{F}B_{|0} - \frac{B}{2}(J + Y)$;

iv) $m(X) + A(U + V - H) + BX = A_{|k}m^k - A(V + H)$;
v) $m(H) + A(Y + E) + \frac{1}{F}X + \frac{1}{2}BH = B_{|k}m^k - \frac{B}{2}(V + H)$.

We note that the assertions of Propositions 4.2 and 4.3 are preserved to changes of local charts.

Now, taking into account (3.9), we have $G^i = \frac{F^2}{2}(Jl^i + Om^i)$. By (3.6) and by Proposition 4.2 i), ii) and vi) result $\dot{\partial}_{\bar{h}}G^i = \frac{F^2}{2}[\bar{m}(O) - \frac{1}{2}\bar{B}O]m^i m_{\bar{h}} = \frac{F}{2}[\bar{A}_{|0} + F\bar{A}(J + Y)]m^i m_{\bar{h}}$. So, we have proved

Lemma 4.1. *For any* 2 *- dimensional complex Finsler space,* $\dot{\partial}_{\bar{h}}G^i = 0$ *if and only if* $\bar{m}(O) = \frac{1}{2}\bar{B}O$*, equivalently with* $\bar{A}_{|0} + F\bar{A}(J + Y) = 0$.

Theorem 4.3. *If* (M, F) *is a Kähler* 2 *- dimensional complex Finsler space, then* $\dot{\partial}_{\bar{h}}G^i = 0$.

Proof. By Propositions 3.2, 4.2 iii) and iv) result $FA[\bar{m}(O) - \frac{1}{2}\bar{B}O] = 0$. So, we have either $A = 0$ or $\bar{m}(O) = \frac{1}{2}\bar{B}O$. If $A = 0$, by Proposition 4.2 vi) we obtain $\bar{m}(O) = \frac{1}{2}\bar{B}O$, which is globally. So that $\dot{\partial}_{\bar{h}}G^i = 0$. If $\bar{m}(O) = \frac{1}{2}\bar{B}O$, by Lemma 4.1, results $\dot{\partial}_{\bar{h}}G^i = 0$. □

Remark 4.1. *The above theorem shows that in dimension two, the class of the complex Berwald spaces coincides with the class of Kähler spaces.*

Theorem 4.4. *A* 2 *- dimensional complex Finsler space is Berwald if and only if it is weakly Kähler and* $\dot{\partial}_{\bar{h}}G^i = 0$.

Proof. The necessity is obvious. For sufficiency, using Propositions 3.2. ii), 4.3 iii) and iv) and Lemma 4.1, it results $\bar{m}(V) - \frac{1}{F}(Y - J) + \frac{1}{2}\bar{B}V = 0$ and $\bar{m}(V) - \frac{1}{F}(E - J) + \frac{1}{2}\bar{B}V = 0$. From here we obtain $Y = E$, i.e. the space is Kähler, and therefore Berwald. □

Proposition 4.4. *If* (M, F) *is a* 2 *- dimensional complex Berwald space, then*

$$\begin{aligned}
U|_{\bar{k}} &= \frac{1}{2F}Ul_{\bar{k}} + [\frac{1}{F}(Y - J) - \frac{1}{2}\bar{B}U]m_{\bar{k}};\ Y|_{\bar{k}} = -\frac{1}{2F}Yl_{\bar{k}} - \frac{1}{F}Om_{\bar{k}};\\
O|_{\bar{k}} &= -\frac{3}{2F}Ol_{\bar{k}} + \frac{1}{2}\bar{B}Om_{\bar{k}};\ X|_{\bar{k}} = \frac{1}{2F}Xl_{\bar{k}} + [\frac{1}{F}(H - 2V) - \bar{B}X]m_{\bar{k}};\\
H|_{\bar{k}} &= \frac{1}{2F}Hl_{\bar{k}} - (\frac{2}{F}Y + \frac{1}{2}\bar{B}H)m_{\bar{k}}; \qquad (4.26)
\end{aligned}$$

equivalently with

$$\begin{aligned}
A_{|\bar{k}} &= -A(\bar{J} + \bar{Y})l_{\bar{k}} - A(\bar{V} + \bar{H})m_{\bar{k}}; \qquad (4.27)\\
B_{|\bar{k}} &= -\frac{B}{2}(\bar{J} + \bar{Y})l_{\bar{k}} - \frac{B}{2}(\bar{V} + \bar{H})m_{\bar{k}};
\end{aligned}$$

equivalently with

$$\begin{aligned} U|_k &= -\frac{1}{2F}Ul_k + [A(Y-J) - \frac{1}{2}BU + \frac{1}{F}X]m_k; \\ Y|_k &= \frac{1}{2F}Yl_k + [\frac{1}{F}(H-U) - AO]m_k; \\ X|_k &= -\frac{3}{2F}Xl_k - [2AU - AH + BX]m_k; \\ H|_k &= -\frac{1}{2F}Hl_k - [2AY + \frac{1}{2}BH + \frac{1}{F}X]m_k; \end{aligned} \tag{4.28}$$

equivalently with

$$\begin{aligned} A_{|k} &= A(J+Y)l_k + A(V+H)m_k; \\ B_{|k} &= \frac{B}{2}(J+Y)l_k + \frac{B}{2}(V+H)m_k. \end{aligned} \tag{4.29}$$

Proof. Under the assumption of Berwald, we have $\dot{\partial}_{\bar{h}}G^i = \dot{\partial}_{\bar{h}}N^i_k = \dot{\partial}_{\bar{h}}L^i_{jk} = 0$ which together with Proposition 4.2 induces (4.26). Using Theorem 2.2 and Propositions 4.2 and 4.3 it results the equivalence between (4.26), (4.27), (4.28) and (4.29). By straightforward computations it results their global validity. □

We note that the equivalent sets of relations (4.26), (4.27), (4.28) and (4.29) have a geometric character and are only necessary conditions for complex Berwald space. These become sufficient together with weakly Kähler condition.

Trivial examples of complex Berwald spaces are given by the purely Hermitian and locally Minkowski manifolds. An nontrivial example of 2 - dimensional complex Berwald space is welcomed.

Let $\Delta = \{(z,w) \in \mathbf{C}^2,\ |w| < |z| < 1\}$ be the Hartogs triangle with the Kähler-purely Hermitian metric

$$a_{i\bar{j}} = \frac{\partial^2}{\partial z^i \partial \overline{z}^j}(\log \frac{1}{(1-|z|^2)\,(|z|^2 - |w|^2)});\ \alpha^2(z,w;\eta,\theta) = a_{i\bar{j}}\eta^i\overline{\eta}^j, \tag{4.30}$$

where $z,\ w, \eta,\ \theta$ are the local coordinates $z^1,\ z^2,\ \eta^1,\ \eta^2$, respectively, and $|z^i|^2 := z^i\bar{z}^i$, $z^i \in \{z,w\}$, $\eta^i \in \{\eta,\theta\}$. We choose

$$b_z = \frac{w}{|z|^2 - |w|^2};\ b_w = -\frac{z}{|z|^2 - |w|^2}. \tag{4.31}$$

With these tools we construct $\alpha(z,w,\eta,\theta) := \sqrt{a_{i\bar{j}}(z,w)\eta^i\overline{\eta}^j}$ and $\beta(z,\eta) = b_i(z,w)\eta^i$ and from here we obtain the complex Randers metric $F = \alpha + |\beta|$

and the complex Kropina metric $F := \frac{\alpha^2}{|\beta|}$. By a direct computation, we deduce

$$\begin{aligned} a_{z\bar{z}} &= \frac{1}{\left(1-|z|^2\right)^2} + b_z b_{\bar{z}};\ a_{z\overline{w}} = b_z b_{\bar{w}};\ a_{w\overline{w}} = b_w b_{\bar{w}}; \\ a^{\bar{z}z} &= \left(1-|z|^2\right)^2;\ a^{\overline{w}z} = \frac{\overline{w}z\left(1-|z|^2\right)^2}{|z|^2}; \\ a^{\overline{w}w} &= \frac{\left(|z|^2-|w|^2\right)^2}{|z|^2} + \frac{|w|^2\left(1-|z|^2\right)^2}{|z|^2}; \\ b^z &= 0;\ b^w = -\frac{|z|^2-|w|^2}{z};\ ||b||^2 = 1;\ \alpha^2 - |\beta|^2 = \frac{|\eta|^2}{\left(1-|z|^2\right)^2} \end{aligned} \tag{4.32}$$

and the horizontal coefficients of the $C-F$ connection are

$$\begin{aligned} L^z_{zz} &= \frac{2\overline{z}}{1-|z|^2};\ L^z_{zw} = L^z_{wz} = 0;\ L^w_{zz} = \frac{1}{1-|z|^2} + \frac{1}{|z|^2-|w|^2}; \\ L^w_{zw} &= L^w_{wz} = -\frac{|z|^2+|w|^2}{z\left(|z|^2-|w|^2\right)};\ L^w_{ww} = \frac{2\overline{w}}{|z|^2-|w|^2}. \end{aligned}$$

which attest the Kähler property. The spray coefficients

$$G^z = \frac{\overline{z}\eta^2}{1-|z|^2};\ G^w = \frac{\overline{z}w}{z}\left(\frac{1}{1-|z|^2} + \frac{1}{|z|^2-|w|^2}\right)\eta^2 + \frac{\overline{w}}{|z|^2-|w|^2}\theta^2$$

are holomorphic in η. Moreover, $K^v_{F,m}(z,\eta) = -\frac{2}{F^2} < 0$.

Proposition 4.5. *If (M,F) is a 2 - dimensional complex Berwald space, then $\mathbf{I}_{|j} = 0$.*

Proof. Firstly, because the space is Berwald, the identity ii) from Proposition 2.2 is $B|_{\bar{k}|j} - B_{|j}|_{\bar{k}} = 0$. On the other hand, the relations (3.7), (3.8), (3.13), (4.1) and (4.29) lead to $B|_{\bar{k}|j} - B_{|j}|_{\bar{k}} = -\mathbf{I}_{|j} m_{\bar{k}}$. So, $\mathbf{I}_{|j} = 0$. □

The converse of the above Proposition is not true. There exist 2 - dimensional complex Finsler spaces with $\mathbf{I}_{|j} = 0$ which are not Berwald. We attest this fact by an example. Namely, we consider the complex version of *Antonelli - Shimada* metric

$$F^2_{AS} = L_{AS}(z,w;\eta,\theta) := e^{2\sigma}\left(|\eta|^4 + |\theta|^4\right)^{\frac{1}{2}},\ \text{with}\ \ \eta,\theta \neq 0, \tag{4.33}$$

on a domain D from $\widetilde{T'M}$, $\dim_C M = 2$, such that its metric tensor is nondegenerated. We relabeled the local coordinates z^1, z^2, η^1, η^2 as z, w, η, θ,

respectively. $\sigma(z,w)$ is a real valued function and $|\eta^i|^2 := \eta^i\overline{\eta}^i$, $\eta^i \in \{\eta, \theta\}$, ([15]).

A direct computation leads to

$$\begin{aligned}
g_{z\overline{z}} &: = g_{1\overline{1}} = \frac{e^{8\sigma}|\eta|^2(|\eta|^4 + 2|\theta|^4)}{L_{AS}^3};\ g^{\overline{z}z} := g^{\overline{1}1} = \frac{2|\eta|^4 + |\theta|^4}{2|\eta|^2 L_{AS}};\\
g_{z\overline{w}} &: = g_{1\overline{2}} = -\frac{e^{8\sigma}|\eta|^2|\theta|^2\overline{\eta}\theta}{L_{AS}^3};\ g^{\overline{w}z} := g^{\overline{2}1} = \frac{\eta\overline{\theta}}{2L_{AS}};\\
g_{w\overline{w}} &: = g_{2\overline{2}} = \frac{e^{8\sigma}|\theta|^2(2|\eta|^4 + |\theta|^4)}{L_{AS}^3};\ g^{\overline{w}w} := g^{\overline{2}2} = \frac{|\eta|^4 + 2|\theta|^4}{2|\theta|^2 L_{AS}};\\
\Delta^2 &= \det\left(g_{i\overline{j}}\right) = \frac{2e^{8\sigma}|\eta|^2|\theta|^2}{L_{AS}^2};\\
l^z &: = l^1 = \frac{\eta}{F_{AS}};\ l^w := l^2 = \frac{\theta}{F_{AS}};\\
l_z &: = l_1 = \frac{e^{4\sigma}|\eta|^2\overline{\eta}}{F_{AS}^3};\ l_w := l_2 = \frac{e^{4\sigma}|\theta|^2\overline{\theta}}{F_{AS}^3};\\
m^z &: = m^1 = -\frac{|\theta|\overline{\theta}}{\sqrt{2}|\eta|F_{AS}};\ m^w := m^2 = \frac{|\eta|\overline{\eta}}{\sqrt{2}|\theta|F_{AS}};\\
m_z &: = m_1 = -\frac{2e^{4\sigma}|\eta||\theta|\theta}{\sqrt{2}F_{AS}^3};\ m_w := m_2 = \frac{2e^{4\sigma}|\eta||\theta|\eta}{\sqrt{2}F_{AS}^3}.
\end{aligned}$$

The nonzero coefficients of the $C - F$ connection are

$$\begin{aligned}
L_{zz}^z &= L_{wz}^w = 2\frac{\partial\sigma}{\partial z};\ L_{zw}^z = L_{ww}^w = 2\frac{\partial\sigma}{\partial w}; \qquad (4.34)\\
C_{zz}^z &= \frac{e^{8\sigma}|\theta|^8\overline{\eta}}{|\eta|^2 L_{AS}^4};\ C_{zw}^z = C_{wz}^z = -\frac{e^{8\sigma}|\theta|^6\overline{\theta}}{L_{AS}^4};\ C_{ww}^z = \frac{e^{8\sigma}|\theta|^4\overline{\theta}^2\eta}{L_{AS}^4};\\
C_{ww}^w &= \frac{e^{8\sigma}|\eta|^8\overline{\theta}}{|\theta|^2 L_{AS}^4};\ C_{zw}^w = C_{wz}^w = -\frac{e^{8\sigma}|\eta|^6\overline{\eta}}{L_{AS}^4};\ C_{zz}^w = \frac{e^{8\sigma}|\eta|^4\overline{\eta}^2\theta}{L_{AS}^4}.
\end{aligned}$$

From here we obtain

$$A = \frac{\overline{\eta}^2\overline{\theta}^2}{2|\eta|^2|\theta|^2 F_{AS}};\ B = \frac{\overline{\eta}\overline{\theta}(|\eta|^4 - |\theta|^4)}{\sqrt{2}|\eta|^3|\theta|^3 F_{AS}} \neq 0;\ \mathbf{I} = \frac{2}{L_{AS}};$$

and so $\mathbf{I}_{|k} = 0$. Moreover, $K_{F_{AS},m}^v(z,\eta) = \frac{4}{L_{AS}} > 0$. The local coefficients L_{jk}^i depend only on z and w, but the Antonelli - Shimada metric is not Berwald because, in generally, it is not Kähler. If σ is a constant, then the Antonelli - Shimada metric is Berwald and locally Minkowski.

The above examples suggest us to pay attention to the class of 2 - dimensional complex Finsler spaces with $\mathbf{I}|_i = -\frac{\mathbf{I}}{F}l_i$ and $\mathbf{I}_{|k} = 0$. With these assumptions we obtain $\frac{\partial \mathbf{I}}{\partial z^k} = N_k^r \mathbf{I}|_r = -\frac{\mathbf{I}}{F} l_r N_k^r = -\frac{\mathbf{I}}{L}\frac{\partial L}{\partial z^k}$. From here it results $L\frac{\partial \mathbf{I}}{\partial z^k} + \mathbf{I}\frac{\partial L}{\partial z^k} = 0$ and so, $\frac{\partial (\mathbf{I}L)}{\partial z^k} = 0$. Thus, we have proved

Theorem 4.5. *Let (M, F) be a connected 2 - dimensional complex Finsler space with $\mathbf{I}_{|k} = 0, \mathbf{I}|_i = -\frac{\mathbf{I}}{F}l_i$ and $AB^2 \neq 0$. Then $\mathbf{I}L$ is a constant on (M, F) and $K^v_{F,m}(z, \eta) = \frac{2c}{L}$, where $c \in \mathbf{R}$.*

Proof. Indeed, from the above considerations we have $\frac{\partial(\mathbf{I}L)}{\partial z^k} = 0$. Therefore, $\mathbf{I}L$ does not depend on z. Hence $\mathbf{I}L = c(\eta, \bar{\eta})$, where $c(\eta, \bar{\eta})$ is real valued. Differentiating, we obtain $\mathbf{I}|_i L + F\mathbf{I}l_i = c|_i$. But, $\mathbf{I}|_i = -\frac{\mathbf{I}}{F}l_i$. Hence $c|_i = 0$ and its conjugate, which means that c is a constant. It results that $\mathbf{I} = \frac{c}{L}$. □

In order to investigate 2 - dimensional complex Landsberg spaces, we translate the $R\Gamma$ and $B\Gamma$ connections in terms of the local complex Berwald frames. After some computations we obtain

$$\begin{aligned} \overset{c}{L}{}^i_{jk} &= Jl^i l_j l_k + \frac{U+V}{2}(l^i m_j l_k + l^i l_j m_k) + [X - \frac{FA}{2}(Y-E)]l^i m_j m_k \\ &\quad + Om^i l_j l_k + \frac{Y+E}{2}(m^i m_j l_k + m^i l_j m_k) + [H - \frac{FB}{2}(Y-E)]m^i m_j m_k \end{aligned} \tag{4.35}$$

and

$$\begin{aligned} \overset{B}{L}{}^i_{jk} &= Jl^i l_j l_k + \frac{U+V}{2}(l^i m_j l_k + l^i l_j m_k) \\ &\quad + \{X + \frac{1}{2}[A_{|0} - FA(J+Y)]\}l^i m_j m_k + Om^i l_j l_k \\ &\quad + \frac{Y+E}{2}(m^i m_j l_k + m^i l_j m_k) + \{H + \frac{1}{2}[B_{|0} - \frac{FB}{2}(J+Y)]\}m^i m_j m_k. \end{aligned} \tag{4.36}$$

By (4.35), (4.36) and (3.11) immediately results

Theorem 4.6. *A 2 - dimensional complex Finsler space is Landsberg if and only if $FA(E-Y) = A_{|0} - FA(J+Y)$ and $FB(E-Y) = B_{|0} - \frac{FB}{2}(J+Y)$.*

Proposition 4.6. *If (M, F) is a 2 - dimensional complex Finsler space weakly Kähler with $B = 0$ then it is Landsberg.*

Proof. Because $U = V$ and $B = 0$, by Proposition 4.1 ii) and Proposition 4.3 ii) we have $FA(E-Y) = A_{|0} - FA(J+Y)$, i.e. the space is Landsberg. □

4.5 The $h\bar{h}-$ Riemann type tensor

Let us investigate the $h\bar{h}-$ Riemann type tensor $R_{\bar{r}j\bar{h}k}$. By (2.5), (3.5) and (3.12) we can write

$R_{\bar{r}j\bar{h}k} = g_{i\bar{r}} R^i_{j\bar{h}k}$
$= -\left(l_i l_{\bar{r}} + m_i m_{\bar{r}}\right)\{(l_{\bar{h}}\bar{\lambda} + m_{\bar{h}}\bar{\mu})(L^i_{jk})$
$+[(l_{\bar{h}}\bar{\lambda} + m_{\bar{h}}\bar{\mu})(N^n_k)](Al^i m_j m_n + Bm^i m_j m_n)\}$
$= -\left(l_i l_{\bar{r}} + m_i m_{\bar{r}}\right) l_{\bar{h}}[\bar{\lambda}(L^i_{jk}) + F\bar{\lambda}(L^n_{sk})l^s(Al^i + Bm^i)m_j m_n]$
$-\left(l_i l_{\bar{r}} + m_i m_{\bar{r}}\right) m_{\bar{h}}[\bar{\mu}(L^i_{jk}) + F\bar{\mu}(L^n_{sk})l^s(Al^i + Bm^i)m_j m_n]$. It results that

$$\begin{aligned} R_{\bar{r}j\bar{h}k} \;=\; & -[\bar{\lambda}(l_i L^i_{jk}) + FA\bar{\lambda}(L^n_{sk}l^s)m_j m_n]l_{\bar{r}}l_{\bar{h}} \\ & -[\bar{\lambda}(L^i_{jk})m_i + FB\bar{\lambda}(L^n_{sk}l^s)m_j m_n]m_{\bar{r}}l_{\bar{h}} \\ & -[\bar{\mu}(l_i L^i_{jk}) + FA\bar{\mu}(L^n_{sk}l^s)m_j m_n]l_{\bar{r}}m_{\bar{h}} \\ & -[\bar{\mu}(L^i_{jk})m_i + FB\bar{\mu}(L^n_{skn}l^s)m_j m]m_{\bar{r}}m_{\bar{h}}. \end{aligned} \tag{4.37}$$

Further on, our goal is to find the link between the horizontal covariant derivatives of the functions (3.10) and theirs properties. Indeed, from (4.37) it follows that $R_{\bar{0}0\bar{h}0} = -LF\bar{\lambda}(l^j l^k l_i L^i_{jk})l_{\bar{h}} - LF\bar{\mu}(l^j l^k l_i L^i_{jk})m_{\bar{h}} = -LJ_{|\bar{0}}l_{\bar{h}} - LFJ_{|\bar{s}}m^{\bar{s}}m_{\bar{h}}$ and $R_{\bar{0}0\bar{0}k} = -LF\bar{\lambda}(l^j l_i L^i_{jk})$. The property $\overline{R_{\bar{0}0\bar{k}0}} = R_{\bar{0}0\bar{0}k}$ leads to $F\bar{\lambda}(l^j l_i L^i_{jk}) = \bar{J}_{|0}l_k + F\bar{J}_{|s}m^s m_k$, which gives

$$\bar{J}_{|0} = J_{|\bar{0}}\ ; \quad \bar{J}_{|s}m^s = \frac{1}{F}V_{|\bar{0}} + \frac{1}{2}V(\bar{J} + \bar{Y}). \tag{4.38}$$

Moreover, by (4.37)
$R_{\bar{r}0\bar{0}0} = -LF\bar{\lambda}(l^j l^k l_i L^i_{jk})l_{\bar{r}} - LF\bar{\lambda}(l^j l^k m_i L^i_{jk})m_{\bar{r}} + \frac{1}{2}LFO(\bar{J} + \bar{Y})m_{\bar{r}}$
$= -LJ_{|\bar{0}}l_{\bar{r}} - LO_{|\bar{0}}m_{\bar{r}} + \frac{1}{2}LFO(\bar{J} + \bar{Y})m_{\bar{r}}$ and
$R_{\bar{0}r\bar{0}0} = -LF\bar{\lambda}(l^k l_i L^i_{rk}) - L^2 A\bar{\lambda}(l^k l^s m_n L^n_{sk})m_r + \frac{1}{2}L^2 AO(\bar{J} + \bar{Y})m_r$
$= -LF\bar{\lambda}(l^k l_i L^i_{rk}) - LFAO_{|\bar{0}}m_r + \frac{1}{2}L^2 AO(\bar{J} + \bar{Y})m_r$.
But, $\overline{R_{\bar{r}0\bar{0}0}} = R_{r\bar{0}0\bar{0}} = R_{\bar{0}r\bar{0}0}$ leads to
$\bar{J}_{|0}l_r + [\bar{O}_{|0} - \frac{1}{2}F\bar{O}(J + Y)]m_r = F\bar{\lambda}(l^k l_i L^i_{rk}) + FA[O_{|\bar{0}} - \frac{1}{2}FO(\bar{J} + \bar{Y})]m_r$.
The contraction with m^r gives

$$\bar{O}_{|0} - \frac{1}{2}F\bar{O}(J + Y) - FAO_{|\bar{0}} + \frac{1}{2}LAO(\bar{J} + \bar{Y}) = U_{|\bar{0}} + \frac{1}{2}FU(\bar{J} + \bar{Y}). \tag{4.39}$$

Next, from (4.37) we have
$R_{\bar{r}0\bar{h}0}m^{\bar{r}} = -L\bar{\lambda}(l^j l^k L^i_{jk})m_i l_{\bar{h}} - L\bar{\mu}(l^j l^k L^i_{jk})m_i m_{\bar{h}}$
$= -F[O_{|\bar{0}}l_{\bar{h}} - \frac{1}{2}FO(\bar{J} + \bar{Y})]l_{\bar{h}} - L[O_{|\bar{s}}m^{\bar{s}} - \frac{1}{2}O(\bar{V} + \bar{H})]m_{\bar{h}}$.
On the other hand
$R_{\bar{0}r\bar{0}h}m^r = -L\bar{\lambda}(m^r l_i L^i_{rh}) - \frac{1}{2}L(\bar{J} + \bar{Y})(Ul_h + Xm_h) - LFA\bar{\lambda}(l^s m_n L^n_{sh})$
$+\frac{1}{2}LFA(\bar{J} + \bar{Y})(Ol_h + Em_h)$.

Using $\overline{R_{\bar{r}0\bar{h}0}m^{\bar{r}}} = R_{r\bar{0}h\bar{0}}m^r = R_{\bar{0}r\bar{0}h}m^r$, we obtain
$[\bar{O}_{|0} - \frac{1}{2}F\bar{O}(J+Y)]l_h + F[\bar{O}_{|s}m^s - \frac{1}{2}\bar{O}(V+H)]m_h$
$= F\bar{\lambda}(m^r l_i L^i_{rh}) + LA\bar{\lambda}(l^s m_n L^n_{sh}) + \frac{1}{2}F(\bar{J}+\bar{Y})(Ul_h + Xm_h)$
$-\frac{1}{2}LA(\bar{J}+\bar{Y})(Ol_h + Em_h)$,
which by transvection with m^h gives

$$\bar{O}_{|s}m^s - \frac{1}{2}\bar{O}(V+H) - AE_{|\bar{0}} = \frac{1}{F}X_{|\bar{0}} + X(\bar{J}+\bar{Y}). \tag{4.40}$$

Taking again into account (4.37), it follows
$R_{\bar{0}0\bar{h}k}m^k = -L\bar{\lambda}(l^j l_i L^i_{jk})m^k l_{\bar{h}} - L\bar{\mu}(l^j l_i L^i_{jk})m^k m_{\bar{h}}$
$= -F[V_{|\bar{0}} + \frac{1}{2}FV(\bar{J}+\bar{Y})]l_{\bar{h}} - L[V_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}V(\bar{V}+\bar{H})]m_{\bar{h}}$ and
$R_{\bar{0}0\bar{k}h}m^{\bar{k}} = -L\bar{\mu}(l^j l_i L^i_{jh})$.
These relations together with $\overline{R_{\bar{0}0\bar{h}k}m^k m^{\bar{h}}} = R_{00\bar{0}h\bar{k}}m^{\bar{k}}m^h = R_{\bar{0}0\bar{k}h}m^{\bar{k}}m^h$ give

$$\bar{V}_{|s}m^s + \frac{1}{2}\bar{V}(V+H) = V_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}V(\bar{V}+\bar{H}). \tag{4.41}$$

Next, (4.37) involves
$R_{\bar{r}0\bar{h}k}m^{\bar{r}}m^k = -F\bar{\lambda}(l^j m^k m_i L^i_{jk})l_{\bar{h}} - F\bar{\mu}(l^j m^k m_i L^i_{jk})m_{\bar{h}}$
$= -E_{|\bar{0}}l_{\bar{h}} - FE_{|\bar{s}}m^{\bar{s}}m_{\bar{h}}$ and
$R_{\bar{0}r\bar{k}h}m^r m^{\bar{k}} = -F\bar{\mu}(l_i L^i_{rh})m^r - LA\bar{\mu}(l^s L^n_{sh})m_n$.
But, $\overline{R_{\bar{r}0\bar{h}k}m^{\bar{r}}m^k} = R_{r\bar{0}h\bar{k}}m^r m^{\bar{k}} = R_{\bar{0}r\bar{k}h}m^r m^{\bar{k}}$ so that
$-\bar{E}_{|0}l_h - F\bar{E}_{|s}m^s m_h = -F\bar{\mu}(l_i L^i_{rh})m^r - LA\bar{\mu}(l^s L^n_{sh})m_n$.
By transvection with l^h and m^h we obtain

$$\begin{aligned} \frac{1}{F}\bar{E}_{|0} - FAO_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}FAO(\bar{V}+\bar{H}) &= U_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}U(\bar{V}+\bar{H}); \\ \bar{E}_{|s}m^s - FAE_{|\bar{s}}m^{\bar{s}} &= X_{|\bar{s}}m^{\bar{s}} + X(\bar{V}+\bar{H}). \end{aligned} \tag{4.42}$$

Using again (4.37), we have
$R_{\bar{r}j\bar{h}k}m^{\bar{r}}m^j m^k = -[\bar{\lambda}(L^i_{jk})m_i m^j m^k + FB\bar{\lambda}(l^s m^k m_n L^n_{sk})]l_{\bar{h}}$
$-[\bar{\mu}(L^i_{jk})m_i m^j m^k + FB\bar{\mu}(l^s m^k m_n L^n_{sk})]m_{\bar{h}}$
$= -(\frac{1}{F}H_{|\bar{0}} + \frac{1}{2}H(\bar{J}+\bar{Y}) + BE_{|\bar{0}})l_{\bar{h}} - (H_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}H(\bar{V}+\bar{H}) + FBE_{|\bar{s}}m^{\bar{s}})m_{\bar{h}}$.
On the other hand,
$R_{\bar{j}r\bar{k}h}m^{\bar{j}}m^r m^{\bar{k}} = -\bar{\mu}(m^r m_i L^i_{rh}) - FB\bar{\mu}(l^s L^n_{sh})m_n$.
But, $\overline{R_{\bar{r}j\bar{h}k}m^{\bar{r}}m^j m^k} = R_{r\bar{j}h\bar{k}}m^{\bar{j}}m^r m^{\bar{k}} = R_{\bar{j}r\bar{k}h}m^{\bar{j}}m^r m^{\bar{k}}$ which leads to
$-(\frac{1}{F}\bar{H}_{|0} + \frac{1}{2}\bar{H}(J+Y) + \bar{B}\bar{E}_{|0})l_h - (\bar{H}_{|s}m^s + \frac{1}{2}\bar{H}(V+H) + F\bar{B}\bar{E}_{|s}m^s)m_h$
$= -\bar{\mu}(m^r m_i L^i_{rh}) - FB\bar{\mu}(l^s L^n_{sh})m_n$.
The transvection with l^h and m^h gives

$$\begin{aligned} \frac{1}{F}\bar{H}_{|0} + \frac{1}{2}\bar{H}(J+Y) + \bar{B}\bar{E}_{|0} &= Y_{|\bar{s}}m^{\bar{s}} + FBO_{|\bar{s}}m^{\bar{s}} - \frac{1}{2}FBO(\bar{V}+\bar{H}); \\ \bar{H}_{|s}m^s + \frac{1}{2}\bar{H}(V+H) + F\bar{B}\bar{E}_{|s}m^s &= H_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}H(\bar{V}+\bar{H}) + FBE_{|\bar{s}}m^{\bar{s}}. \end{aligned}$$

Now, $R_{\bar{r}j\bar{h}0}m^{\bar{r}}m^{j} = -[F\bar{\lambda}(m^{j}l^{k}m_{i}L^{i}_{jk}) + LB\bar{\lambda}(l^{s}l^{k}L^{n}_{sk})m_{n}]l_{\bar{h}}$
$-[F\bar{\mu}(m^{j}l^{k}m_{i}L^{i}_{jk}) + LB\bar{\mu}(l^{s}l^{k}L^{n}_{sk})m_{n}]m_{\bar{h}}$
$= -[Y_{|\bar{0}} + FBO_{|\bar{0}} - \frac{1}{2}LO(\bar{J}+\bar{Y})]l_{\bar{h}}$
$-F[Y_{|\bar{s}}m^{\bar{s}} + FBO_{|\bar{s}}m^{\bar{s}} - \frac{1}{2}FBO(\bar{V}+\bar{H})]m_{\bar{h}}$
and
$R_{\bar{j}r\bar{0}h}m^{\bar{j}}m^{r} = -F\bar{\lambda}(m^{r}m_{i}L^{i}_{rh}) - LB\bar{\lambda}(l^{s}L^{n}_{sh})m_{n}.$
The conjugation $\overline{R_{\bar{r}j\bar{h}0}m^{\bar{r}}m^{j}l^{\bar{h}}} = R_{r\bar{j}h\bar{0}}m^{\bar{j}}m^{r}l^{h} = R_{\bar{j}r\bar{0}h}m^{\bar{j}}m^{r}l^{h}$ gives

$$\bar{Y}_{|0} + F\bar{B}\bar{O}_{|0} - \frac{1}{2}L\bar{O}(J+Y) = Y_{|\bar{0}} + FBO_{|\bar{0}} - \frac{1}{2}LO(\bar{J}+\bar{Y}). \tag{4.43}$$

Lemma 4.2. *Let (M,F) be a 2 - dimensional weakly Kähler complex Finsler space. Then*
i) $\frac{1}{F}\bar{O}_{|0} - \frac{1}{2}\bar{O}(J+Y) - AO_{|\bar{0}} + \frac{1}{2}FAO(\bar{J}+\bar{Y}) = \bar{J}_{|s}m^{s}$;
ii) $\frac{1}{F}\bar{E}_{|0} - FAO_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}FAO(\bar{V}+\bar{H}) = V_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}V(\bar{V}+\bar{H})$.

Proof. It results by Proposition 3.2, (4.38), (4.39) and (4.42) . By computation using (3.11), we obtain the global validity of these assertions. □

Remark 4.2. *If (M,F) is purely Hermitian ($A=0$) and Kähler, then* $\frac{1}{F}\bar{O}_{|0} - \frac{1}{2}\bar{O}(J+Y) = \bar{J}_{|s}m^{s}$ *and* $V_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}V(\bar{V}+\bar{H}) = \frac{1}{F}\bar{E}_{|0}$.

In order to show the geometrical aspects of the above computations, considering (2.7), we define *the horizontal holomorphic sectional curvature* in direction λ by

$$K^{h}_{F,\lambda}(z,\eta) := 2\mathbf{R}(\lambda,\bar{\lambda},\lambda,\bar{\lambda}) \tag{4.44}$$

and *the horizontal holomorphic sectional curvature* in direction μ by

$$K^{h}_{F,\mu}(z,\eta) = 2\mathbf{R}(\mu,\bar{\mu},\mu,\bar{\mu}). \tag{4.45}$$

Theorem 4.7. *Let (M,F) be a 2 - dimensional complex Finsler space. Then*
i) $K^{h}_{F,\lambda}(z,\eta) = 2\mathbf{K}$, *where* $\mathbf{K} := -\frac{1}{F}J_{|\bar{0}}$;
ii) $K^{h}_{F,\mu}(z,\eta) = 2\mathbf{W}$, *where* $\mathbf{W} := -H_{|\bar{s}}m^{\bar{s}} - \frac{1}{2}H(\bar{V}+\bar{H}) - BFE_{|\bar{s}}m^{\bar{s}}$.

Proof. By (4.37) we obtain $\mathbf{R}(\lambda,\bar{\lambda},\lambda,\bar{\lambda}) = l^{\bar{h}}l^{\bar{r}}l^{j}l^{k}R_{\bar{r}j\bar{h}k} = -\bar{\lambda}(l^{j}l^{k}l_{i}L^{i}_{jk}) = -\frac{1}{F}J_{|\bar{0}}$ and so *i)* is proved. Similarly, we have
$\mathbf{R}(\mu,\bar{\mu},\mu,\bar{\mu}) = m^{\bar{h}}m^{\bar{r}}m^{j}m^{k}R_{\bar{r}j\bar{h}k} = -\bar{\mu}(L^{i}_{jk})m^{j}m^{k}m_{i} - FB\bar{\mu}(l^{s}L^{n}_{sk})m^{k}m_{n}$
$= -\bar{\mu}(H) - \frac{1}{2}H(\bar{V}+\bar{H}) - FB\bar{\mu}(E) = -H_{|\bar{s}}m^{\bar{s}} - \frac{1}{2}H(\bar{V}+\bar{H}) - FBE_{|\bar{s}}m^{\bar{s}}$,
i.e. *ii)*.

Changing the local coordinates $(z^{k},\eta^{k})_{k=\overline{1,2}}$ into $(z'^{k},\eta'^{k})_{k=\overline{1,2}}$, it results $\mathbf{K}' = \mathbf{K}$ and $\mathbf{W}' = \mathbf{W}$, which complete the proof. □

We call the functions $\mathbf{K}$ and $\mathbf{W}$ the *horizontal curvature invariants*. Further on, our goal is to find the link between the $h\bar{h}-$ Riemann type tensors $R_{\bar{r}j\bar{h}k}$, $\mathbf{K}$ and $\mathbf{W}$.

Then, using (2.5), $\delta_i = l_i\lambda + m_i\mu$ and (4.37), $R_{\bar{r}j\bar{h}k} = \mathbf{R}(\delta_j, \delta_{\bar{r}}, \delta_k, \delta_{\bar{h}})$ is decomposed into sixteen terms.

Proposition 4.7. *Let (M, F) be a 2 - dimensional complex Finsler space. Then*

$$\begin{aligned}
R_{\bar{r}j\bar{h}k} &= \mathbf{K}l_{\bar{r}}l_jl_{\bar{h}}l_k + \mathbf{W}m_{\bar{r}}m_jm_{\bar{h}}m_k \qquad (4.46)\\
&-[\frac{1}{F}\bar{O}_{|0} - \frac{1}{2}\bar{O}(J+Y)]l_{\bar{r}}m_jl_{\bar{h}}l_k - [\frac{1}{F}O_{|\bar{0}} - \frac{1}{2}O(\bar{J}+\bar{Y})]m_{\bar{r}}l_jl_{\bar{h}}l_k\\
&-\bar{J}_{|s}m^sl_{\bar{r}}l_jl_{\bar{h}}m_k - J_{|\bar{s}}m^{\bar{s}}l_{\bar{r}}l_jm_{\bar{h}}l_k\\
&-[V_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}V(\bar{V}+\bar{H})]l_{\bar{r}}l_jm_{\bar{h}}m_k - \frac{1}{F}\bar{E}_{|0}m_{\bar{r}}l_jl_{\bar{h}}m_k\\
&-\frac{1}{F}E_{|\bar{0}}l_{\bar{r}}m_jm_{\bar{h}}l_k - [\frac{1}{F}Y_{|\bar{0}} + BO_{|\bar{0}} - \frac{1}{2}FBO(\bar{J}+\bar{Y})]m_{\bar{r}}m_jl_{\bar{h}}l_k\\
&-E_{|\bar{s}}m^{\bar{s}}m_{\bar{r}}l_jm_{\bar{h}}m_k - [\frac{1}{F}\bar{H}_{|0} + \frac{1}{2}\bar{H}(J+Y) + \bar{B}\bar{E}_{|0}]m_{\bar{r}}m_jm_{\bar{h}}l_k\\
&-\bar{E}_{|s}m^sl_{\bar{r}}m_jm_{\bar{h}}m_k - [\frac{1}{F}H_{|\bar{0}} + \frac{1}{2}H(\bar{J}+\bar{Y}) + BE_{|\bar{0}}]m_{\bar{r}}m_jl_{\bar{h}}m_k\\
&-[O_{|\bar{s}}m^{\bar{s}} - \frac{1}{2}O(\bar{V}+\bar{H})]m_{\bar{r}}l_jm_{\bar{h}}l_k - [\bar{O}_{|s}m^s - \frac{1}{2}\bar{O}(V+H)]l_{\bar{r}}m_jl_{\bar{h}}m_k.
\end{aligned}$$

Remark 4.3. *If $R_{\bar{r}j\bar{h}k} = 0$ then the horizontal holomorphic sectional curvature in any direction is zero.*

Now we come back to the Antonelli-Shimada metric (4.33) in order to study its horizontal curvature invariants $\mathbf{K}$ and $\mathbf{W}$ and its $h\bar{h}-$ Riemann type tensor. After some direct computations we get

$$\begin{aligned}
J &= Y = 2\frac{\partial\sigma}{\partial z^i}\eta^i;\ E = O = 0;\\
H &= V = -\frac{\sqrt{2}}{|\eta||\theta|F_{AS}}\left(\frac{\partial\sigma}{\partial z}|\theta|^2\overline{\theta} - \frac{\partial\sigma}{\partial w}|\eta|^2\overline{\eta}\right),
\end{aligned}$$

which substituted into (4.46) give

$$R_{\bar{r}j\bar{h}k} = g_{j\bar{r}}(\mathbf{K}l_{\overline{h}}l_k + \mathbf{W}m_{\overline{h}}m_k - \bar{J}_{|s}m^sl_{\bar{h}}m_k - J_{|\bar{s}}m^{\bar{s}}m_{\bar{h}}l_k),$$

where

$$\begin{aligned}
\mathbf{K} &= -\frac{2}{L_{AS}}\frac{\partial^2\sigma}{\partial z^k\partial\overline{z}^h}\eta^k\overline{\eta}^h; \qquad (4.47)\\
\mathbf{W} &= -\frac{\mathbf{K}}{2} - \frac{e^{-4\sigma}L_{AS}}{|\eta|^2|\theta|^2}\left(\frac{\partial^2\sigma}{\partial z\partial\overline{z}}|\theta|^2 + \frac{\partial^2\sigma}{\partial w\partial\overline{w}}|\eta|^2\right).
\end{aligned}$$

Proposition 4.8. *The horizontal holomorphic sectional curvature in direction* λ *of the Antonelli-Shimada metric,* $K^h_{F_{AS},\lambda}(z,\eta)$ *is strictly negative (positive) if and only if the* $(1,1)-$ *form* $\frac{\partial^2\sigma}{\partial z^k\partial\bar{z}^h}\eta^k\overline{\eta}^h$ *is positive (negative) definite.*

Proof. Indeed, $K^h_{F_{AS},\lambda}(z,\eta) = 2\mathbf{K} = -\frac{4}{L_{AS}}\frac{\partial^2\sigma}{\partial z^k\partial\bar{z}^h}\eta^k\overline{\eta}^h$. Its sign depends on the sign of the $(1,1)-$ form $\frac{\partial^2\sigma}{\partial z^k\partial\bar{z}^h}\eta^k\overline{\eta}^h$. $\square$

For example if $\sigma(z,w) = \log\frac{1}{(1-|z|^2)(|z|^2-|w|^2)}$ then $\frac{\partial^2\sigma}{\partial z^k\partial\bar{z}^h}\eta^k\overline{\eta}^h$ is a purely Hermitian metric on the Hartogs triangle $D = \{(z,w) \in \mathbf{C}^2,\ |w| < |z| < 1\}$. Therefore, $K^h_{F_{AS},\lambda}(z,\eta) < 0$.

Another example, if $\sigma(z,w) = \log(1-|z|^2-|w|^2)$, (it) leads to the Bergman metric $-\frac{\partial^2\sigma}{\partial z^k\partial\bar{z}^h}\eta^k\overline{\eta}^h$ on the unit disk $D^2 := \{(z,w) \in \mathbf{C}^2,\ |z|^2+|w|^2 < 1\}$. It results that $K^h_{F_{AS},\lambda}(z,\eta) > 0$ and $K^h_{F_{AS},\mu}(z,\eta) < 0$.

Proposition 4.9. *If* $\sigma(z,w)$ *is a harmonic function, i.e.* $\frac{\partial^2\sigma}{\partial z\partial\bar{z}} = \frac{\partial^2\sigma}{\partial w\partial\overline{w}} = 0$, *then* $K^h_{F_{AS},\mu}(z,\eta) = -\frac{1}{2}K^h_{F_{AS},\lambda}(z,\eta)$.

Proof. It results by (4.47). $\square$

We point out that the $h\bar{h}-$ Riemann type tensors $R_{\bar{r}j\bar{h}k}$ generally are not symmetric. But, (4.46) permits us to study this particular case and some others.

4.5.1 A weakly symmetry condition

We call the property $R_{\bar{0}k\bar{0}0} = R_{\bar{0}0\bar{0}k}$ as being *a weakly symmetry* condition of the curvature. First, we want to see what does this condition mean, in terms of the horizontal terms (3.10). The answer is below.

Corollary 4.1. *Let* (M,F) *be a 2 - dimensional complex Finsler space. Then* $R_{\bar{0}k\bar{0}0} = R_{\bar{0}0\bar{0}k}$ *if and only if* $\bar{J}_{|s}m^s = \frac{1}{F}\bar{O}_{|0} - \frac{1}{2}\bar{O}(J+Y)$.

Proof. (4.46) gives that $R_{\bar{0}k\bar{0}0} = F^3\{\mathbf{K}l_k - [\frac{1}{F}\bar{O}_{|0} - \frac{1}{2}\bar{O}(J+Y)]m_k\}$ and $R_{\bar{0}0\bar{0}k} = F^3[\mathbf{K}l_k - \bar{J}_{|s}m^s m_k]$. So (that), $R_{\bar{0}k\bar{0}0} = R_{\bar{0}0\bar{0}k}$ iff $\bar{J}_{|s}m^s = \frac{1}{F}\bar{O}_{|0} - \frac{1}{2}\bar{O}(J+Y)$, which is globally by (3.11). $\square$

Proposition 4.10. *Let* (M,F) *be a 2 - dimensional weakly Kähler complex Finsler space with* $R_{\bar{0}k\bar{0}0} = R_{\bar{0}0\bar{0}k}$. *Then it is either purely Hermitian or with* $J_{|\bar{s}}m^{\bar{s}} = 0$.

Proof. Lemma 4.2 i) together with Corollary 4.1 leads to $AJ_{|\bar{s}}m^{\bar{s}} = 0$, which gives $A = 0$ or $J_{|\bar{s}}m^{\bar{s}} = 0$. Both conditions work globally. $\square$

Further on, our goal is to find the properties of the invariants $\mathbf{K}$ and $\mathbf{W}$ with this condition of weakly symmetry. Therefore, we write the identity i) of Proposition 2.2 for J

$$J|_{\bar{r}|\bar{s}} - J_{|\bar{s}}|_{\bar{r}} = C^{\bar{n}}_{\bar{s}\bar{r}} J_{|\bar{n}}. \tag{4.48}$$

Using 4.2 i) and (3.13) we have

$$J|_{\bar{r}|\bar{s}} = -\frac{1}{2F} J_{|\bar{s}} l_{\bar{r}} + [\frac{1}{F} O_{|\bar{s}} - \frac{1}{2F} O(\bar{J}+\bar{Y}) l_{\bar{s}} - \frac{1}{2F} O(\bar{V}+\bar{H}) m_{\bar{s}}] m_{\bar{r}}. \tag{4.49}$$

On the other hand, $J_{|\bar{s}} = -\mathbf{K} l_{\bar{s}} + J_{|\bar{h}} m^{\bar{h}} m_{\bar{s}}$ and using (3.7) we obtain

$$J_{|\bar{s}}|_{\bar{r}} = -\left(\mathbf{K}|_{\bar{r}} + \frac{1}{2F}\mathbf{K} l_{\bar{r}} + \frac{1}{F} J_{|\bar{r}}\right) l_{\bar{s}} + J_{|\bar{h}}|_{\bar{r}} m^{\bar{h}} m_{\bar{s}}. \tag{4.50}$$

Plugging (4.49) and (4.50) into (4.48), it results
$-\frac{1}{2F} J_{|\bar{s}} l_{\bar{r}} + \frac{1}{F} O_{|\bar{s}} m_{\bar{r}} + \left(\mathbf{K}|_{\bar{r}} + \frac{1}{2F}\mathbf{K} l_{\bar{r}} + \frac{1}{F} J_{|\bar{r}} - \frac{1}{2F} O(\bar{J}+\bar{Y}) m_{\bar{r}}\right) l_{\bar{s}}$
$-\left(J_{|\bar{h}}|_{\bar{r}} m^{\bar{h}} + \frac{1}{2F} O(\bar{V}+\bar{H}) m_{\bar{r}}\right) m_{\bar{s}} = (-\bar{A}\mathbf{K} + \bar{B} J_{|\bar{h}} m^{\bar{h}}) m_{\bar{r}} m_{\bar{s}}$
which contracted by $l^{\bar{s}}$ and $m^{\bar{s}} m^{\bar{r}}$ respectively, leads to

$$\begin{aligned} \mathbf{K}|_{\bar{r}} &= -\frac{1}{F}[J_{|\bar{h}} m^{\bar{h}} + \frac{1}{F} O_{|\bar{0}} - \frac{1}{2} O(\bar{J}+\bar{Y})] m_{\bar{r}}; \\ \bar{A}\mathbf{K} &= J_{|\bar{h}}|_{\bar{r}} m^{\bar{h}} m^{\bar{r}} + \bar{B} J_{|\bar{h}} m^{\bar{h}} - \frac{1}{F} O_{|\bar{s}} m^{\bar{s}} + \frac{1}{2F} O(\bar{V}+\bar{H}). \end{aligned} \tag{4.51}$$

Transvecting the Bianchi identity

$$\mathcal{A}_{kl}\left\{R_{\bar{r}j\bar{h}k|l} - P_{\bar{r}j\bar{s}k} R^{\bar{s}}_{\bar{0}l\bar{h}}\right\} + R_{\bar{r}j\bar{h}n} T^n_{kl} = 0, \tag{4.52}$$

(see [15], p. 77), by $\bar{\eta}^r \eta^j \bar{\eta}^h \eta^k$ it follows

$$F\mathbf{K}_{|l} - \mathbf{K}_{|0} l_l - \bar{J}_{|s|0} m^s m_l + F\mathbf{K} l^k l_n T^n_{kl} - F\bar{J}_{|s} m^s l^k m_n T^n_{kl} = 0. \tag{4.53}$$

Theorem 4.8. *Let (M,F) be a connected 2 - dimensional weakly Kähler complex Finsler space with $R_{\bar{0}k\bar{0}0} = R_{\bar{0}0\bar{0}k}$ and $|A| \neq 0$. Then $\mathbf{K}$ is a constant on (M,F).*

Proof. By Proposition 4.10 and by the first relation of (4.51) it results that $\mathbf{K}|_{\bar{r}} = 0$, i.e. $\mathbf{K}$ does not depend on η. Because (M,F) is weakly Kähler, the identity (4.53) together with $\bar{J}_{|s|0} m^s = 0$ gives $\mathbf{K}_{|l} = \frac{1}{F}\mathbf{K}_{|0} l_l$. But, using ii) from Proposition 2.2, we have $0 = \mathbf{K}|_{\bar{k}|j} = \mathbf{K}_{|j}|_{\bar{k}}$. On the other hand,
$\mathbf{K}_{|l}|_{\bar{r}} = -\frac{1}{2L}\mathbf{K}_{|0} l_l l_{\bar{r}} + \frac{1}{F}\mathbf{K}_{|0}|_{\bar{r}} l_l + \frac{1}{F}\mathbf{K}_{|0}(\frac{1}{2F} l_l l_{\bar{r}} + \frac{1}{F} m_l m_{\bar{r}})$.
It follows that $\mathbf{K}_{|0} = 0$ and so $\mathbf{K}_{|l} = 0$, which is equivalent to $\frac{\partial \mathbf{K}}{\partial z^l} = 0$, i.e. $\mathbf{K}$ is a constant on (M,F). □

Corollary 4.2. *Let (M,F) be a connected 2 - dimensional complex Finsler space with $R_{\bar{0}k\bar{0}0}=R_{\bar{0}0\bar{0}k}$. Then* $\mathbf{K}$ *depends on z only if and only if $\bar{J}_{|s}m^s=0$. Moreover, given any of these equivalent conditions, we have $F\bar{A}\mathbf{K}=-O_{|\bar{s}}m^{\bar{s}}+\frac{1}{2}O(\bar{V}+\bar{H})$.*

Proof. By Corollary 4.1, the first relation of (4.51) is $\mathbf{K}|_{\bar{r}}=-\frac{2}{F}J_{|\bar{h}}m^{\bar{h}}m_{\bar{r}}$ which justifies the above equivalence. $\bar{J}_{|s}m^s=0$ with the second equation of (4.51) give $F\bar{A}\mathbf{K}=-O_{|\bar{s}}m^{\bar{s}}+\frac{1}{2}O(\bar{V}+\bar{H})$. Its global validity completes the proof. □

Theorem 4.9. *Let (M,F) be a connected 2 - dimensional complex Finsler space with $R_{\bar{0}k\bar{0}0}=R_{\bar{0}0\bar{0}k}$, $|A|\neq 0$ and* $\mathbf{K}$ *a nonzero constant on M. Then (M,F) is weakly Kähler.*

Proof. Substituting $\mathbf{K}_{|l}=\bar{J}_{|s}m^s=0$ in the relation (4.53) it follows that $L\mathbf{K}l^kl_nT^n_{kl}=0$. Consequently, $l^kl_nT^n_{kl}=0$, since $\mathbf{K}\neq 0$. □

The purely Hermitian case is characterized by

Proposition 4.11. *If (M,F) is a 2 - dimensional Kähler purely Hermitian space, then $R_{\bar{0}k\bar{0}0}=R_{\bar{0}0\bar{0}k}$.*

Proof. It results by Remark 4.2. □

Theorem 4.10. *Let (M,F) be a connected 2 - dimensional Kähler purely Hermitian space. Then* $\mathbf{K}$ *is a constant on (M,F) if and only if $J_{|\bar{h}}m^{\bar{h}}=0$.*

Proof. By (4.51) we have $\mathbf{K}|_{\bar{r}}=-\frac{2}{F}J_{|\bar{h}}m^{\bar{h}}m_{\bar{r}}$. If $\mathbf{K}$ is a constant on (M,F) then $J_{|\bar{h}}m^{\bar{h}}=0$. Conversely, if $J_{|\bar{h}}m^{\bar{h}}=0$ then $\mathbf{K}|_{\bar{r}}=0$ and by same arguments as in Theorem 4.8 it results that $\mathbf{K}$ is a constant on (M,F). □

We note that the above Theorems give the necessary and sufficient conditions that a connected 2 - dimensional complex Finsler space should be of constant horizontal holomorphic curvature in direction λ. It is interesting for us to see what happens with the horizontal holomorphic curvature in direction μ, in this case.

Proposition 4.12. *Let (M,F) be a connected 2 - dimensional weakly Kähler complex Finsler space with $R_{\bar{0}k\bar{0}0}=R_{\bar{0}0\bar{0}k}$ and $|A|\neq 0$. Then*

i) $F^2A\mathbf{K}=\Phi_{|0}-F(J+Y)\Phi$;
ii) $E_{|\bar{0}}=-\frac{F\mathbf{K}}{2}\left(1+A\bar{A}L\right)$;
iii) $V_{|\bar{s}}m^{\bar{s}}+\frac{1}{2}V(\bar{V}+\bar{H})=-\frac{\mathbf{K}}{2}\left(1-A\bar{A}L\right)$;
iv) $Y_{|\bar{0}}=-\frac{F\mathbf{K}}{2}\left(1-A\bar{A}L\right)+F\left|\Phi\right|^2$;
v) $H_{|\bar{0}}+\frac{F}{2}H(\bar{J}+\bar{Y})=-\frac{ALF\mathbf{K}}{2}\left(\bar{B}-F\bar{A}B\right)+\frac{\bar{A}L}{2}[\Phi_{|k}m^k-(V+H)\Phi]+L\Phi\bar{\Omega}$;

vi) $E_{|\bar{s}}m^{\bar{s}} = -\frac{F\mathbf{K}}{2}\left(\bar{B} - F\bar{A}B\right) - \frac{AF}{2}[\bar{\Phi}_{|\bar{k}}m^{\bar{k}} - (\bar{V} + \bar{H})\bar{\Phi}]$;
vii) $\mathbf{W} = \mathbf{K}\left(1 + A\bar{A}L\right) - F(E_{|\bar{s}}m^{\bar{s}})|_l m^l - \frac{3}{2}BFE_{|\bar{s}}m^{\bar{s}} - L\left|\Omega\right|^2$,
where $\Phi := A_{|\bar{0}} + AF(\bar{J} + \bar{Y})$ *and* $\Omega := A_{|\bar{k}}m^{\bar{k}} + A(\bar{V} + \bar{H})$.

Proof. Let us consider the Bianchi identity, (see [15], p. 77),

$$R_{\bar{r}j\bar{h}k}|_l - \Xi_{\bar{r}j\bar{h}l|k} - P_{\bar{r}j\bar{s}k}P^{\bar{s}}_{\bar{0}l\bar{h}} + S_{\bar{r}j\bar{s}l}R^{\bar{s}}_{\bar{0}k\bar{h}} + R_{\bar{r}j\bar{h}n}C^n_{kl} = 0. \tag{4.54}$$

In order to prove the statements i)-vii), we use Theorem 4.8, the covariant derivatives (3.7), (3.13) and the expressions of the $v\bar{v}-$, $h\bar{v}-$, $v\bar{h}-$, $h\bar{h}-$ Riemann type tensors.

Contracting into (4.54) by $\bar{\eta}^r m^j \bar{\eta}^h \eta^k$, using
$R_{\bar{r}j\bar{h}k}|_l\bar{\eta}^r m^j\bar{\eta}^h\eta^k = -R_{\bar{0}j\bar{0}l}m^j = F^2[\bar{O}_{|s}m^s - \frac{1}{2}\bar{O}(V+H)]m_l = -F^3A\mathbf{K}m_l$;
$P_{\bar{r}j\bar{s}k}\bar{\eta}^r = S_{\bar{r}j\bar{s}l}\bar{\eta}^r = C^n_{kl}\eta^k = 0$ and
$\Xi_{\bar{r}j\bar{h}l|k}\bar{\eta}^r m^j\bar{\eta}^h\eta^k = -F[\Phi_{|0} - F(J+Y)\Phi]m_l$, we obtain i).

The contraction with $\bar{\eta}^r\eta^j\bar{m}^h\eta^k$ of (4.54),
$R_{\bar{r}j\bar{h}k}|_l\bar{\eta}^r\eta^j\bar{m}^h\eta^k = -R_{\bar{0}l\bar{h}0}\bar{m}^h - R_{\bar{0}0\bar{h}l}\bar{m}^h + \frac{1}{L}R_{\bar{0}0\bar{0}0}m_l$
$= F^2[\frac{1}{F}\bar{E}_{|0} + V_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}V(\bar{V} + \bar{H}) + \mathbf{K}]m_l$ and $\Xi_{\bar{r}j\bar{h}k}\eta^j = 0$ lead to

$$\frac{1}{F}\bar{E}_{|0} + V_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}V(\bar{V} + \bar{H}) = -\mathbf{K}.$$

On the other hand, by Lemma 4.2 ii),

$$\frac{1}{F}\bar{E}_{|0} - V_{|\bar{s}}m^{\bar{s}} - \frac{1}{2}V(\bar{V} + \bar{H}) = -LA\bar{A}\mathbf{K}.$$

The last two relations give ii) and iii).

Now, contracting again (4.54) by $\bar{m}^r\eta^j\bar{\eta}^h\eta^k$, we have
$R_{\bar{r}j\bar{h}k}|_l\bar{m}^r\eta^j\bar{\eta}^h\eta^k = F^2(\mathbf{K} + \frac{1}{F}Y_{|\bar{0}} + \frac{1}{F}E_{|\bar{0}})m_l$ and
$P_{\bar{r}j\bar{s}k}P^{\bar{s}}_{\bar{0}l\bar{h}}\bar{m}^r\eta^j\bar{\eta}^h\eta^k = F^2\left|\Phi\right|^2 m_l$.

It results $[\mathbf{K} + \frac{1}{F}Y_{|\bar{0}} + \frac{1}{F}E_{|\bar{0}} - \left|\Phi\right|^2]m_l = 0$. Hereby, $Y_{|\bar{0}} = -\mathbf{K}F - E_{|\bar{0}} + F\left|\Phi\right|^2$, which together with ii) implies iv).

Next we prove v) and vi). First we contract (4.54) with $\bar{\eta}^r m^j\bar{\eta}^h m^k m^l$ and we obtain
$(R_{\bar{0}j\bar{0}k}m^j m^k)|_l m^l - BR_{\bar{0}j\bar{0}k}m^j m^k - \Xi_{\bar{0}j\bar{0}l|k}m^j m^k m^l + R_{\bar{0}j\bar{0}n}C^n_{kl}m^j m^k m^l = 0$.
This implies that

$$A|_l m^l L\mathbf{K} = -[\Phi_{|k}m^k + (V + H)\Phi] \tag{4.55}$$

The contraction of (4.54) by $\bar{m}^r\eta^j\bar{\eta}^h m^k m^l$ implies
$(R_{\bar{r}0\bar{0}k}\bar{m}^r m^k)|_l m^l - (R_{\bar{r}l\bar{0}k}\bar{m}^r m^k + P_{\bar{r}0\bar{s}k}P^{\bar{s}}_{\bar{0}l\bar{0}}\bar{m}^r m^k - R_{\bar{r}0\bar{0}n}C^n_{kl}\bar{m}^r m^k)m^l = 0$,
which gives

$H_{|\bar{0}} + \frac{F}{2}H(\bar{J}+\bar{Y}) = F\bar{E}_{|0}|_l m^l + L\Phi\bar{\Omega}$. Now, this together with ii), (4.55) and (4.1) gives v).

The contraction of (4.54) by $\bar{m}^r\eta^j\bar{m}^h\eta^k m^l$ gives
$(R_{\bar{r}0\bar{h}0}\bar{m}^r\bar{m}^h)|_l m^l + BR_{\bar{r}0\bar{h}0}\bar{m}^r\bar{m}^h - R_{\bar{r}l\bar{h}0}\bar{m}^r\bar{m}^h m^l - R_{\bar{r}0\bar{h}l}\bar{m}^r\bar{m}^h m^l$
$-P_{\bar{r}0\bar{s}0}P^{\bar{s}}_{\bar{0}l\bar{h}}\bar{m}^r\bar{m}^h m^l = 0$, which is equivalent to
$L\mathbf{K}\bar{A}|_l m^l + \frac{1}{F}\bar{H}_{|0} + \frac{1}{2}\bar{H}(J+Y) + \bar{B}\bar{E}_{|0} + E_{|\bar{s}}m^{\bar{s}} + B\bar{A}L\mathbf{K} = F\bar{\Phi}\Omega$.
Using ii), v) and (4.1) it leads to vi).

For vii) we contract (4.54) with $\bar{m}^r\eta^j\bar{m}^h m^k m^l$ and we deduce
$(R_{\bar{r}0\bar{h}k}\bar{m}^r\bar{m}^h m^k)|_l m^l + \frac{B}{2}R_{\bar{r}0\bar{h}k}\bar{m}^r\bar{m}^h m^k$
$+\frac{1}{L}R_{\bar{0}0\bar{h}k}\bar{m}^h m^k - R_{\bar{r}l\bar{h}k}\bar{m}^r\bar{m}^h m^k m^l + \frac{1}{L}R_{\bar{r}0\bar{0}k}\bar{m}^r m^k$
$-P_{\bar{r}0\bar{s}k}P^{\bar{s}}_{\bar{0}l\bar{h}}\bar{m}^r\bar{m}^h m^k m^l + R_{\bar{r}0\bar{h}n}C^n_{kl}\bar{m}^r\bar{m}^h m^k m^l = 0$.
From here we obtain
$-F(E_{|\bar{s}}m^{\bar{s}})|_l m^l - \frac{B}{2}E_{|\bar{s}}m^{\bar{s}} - V_{|\bar{s}}m^{\bar{s}} - \frac{1}{2}V(\bar{V}+\bar{H}) - \mathbf{W} - \frac{1}{F}\bar{E}_{|0} + A\bar{A}L\mathbf{K} - BFE_{|\bar{s}}m^{\bar{s}} = L|\Omega|^2$, which leads to vii). The global validity of the above statements results by straightforward computations using (3.11). □

Next, we establish some consequences of the above Proposition. From vii) and Theorem 4.7 ii) it immediately results

Corollary 4.3. *Let (M,F) be a connected 2 - dimensional weakly Kähler complex Finsler space with $R_{\bar{0}k\bar{0}0} = R_{\bar{0}0\bar{0}k}$ and $|A| \neq 0$. Then*

$$K^h_{F,\mu}(z,\eta) = 2\mathbf{K}\left(1 + A\bar{A}L\right) - 2F(E_{|\bar{s}}m^{\bar{s}})|_l m^l - 3BFE_{|\bar{s}}m^{\bar{s}} - 2L|\Omega|^2.$$

So, we remark that the conditions which assure that $K^h_{F,\lambda}(z,\eta)$ is a constant on M do not suffice to imply that $K^h_{F,\mu}(z,\eta)$ is a constant, too.

Proposition 4.12 i) gives

Corollary 4.4. *Let (M,F) be a connected 2 - dimensional weakly Kähler complex Finsler space with $R_{\bar{0}k\bar{0}0} = R_{\bar{0}0\bar{0}k}$ and $|A| \neq 0$. Then $\mathbf{K} = 0$ if and only if $\Phi_{|0} = F(J+Y)\Phi$.*

Corollary 4.5. *Let (M,F) be a connected 2 - dimensional weakly Kähler complex Finsler space with $R_{\bar{0}k\bar{0}0} = R_{\bar{0}0\bar{0}k}$ and $|A| \neq 0$. If $\Phi_{|k}m^k = (V+H)\Phi$ then*

$$\mathbf{W} = \mathbf{K}\left(1 + A\bar{A}L\right) - \frac{L\mathbf{K}}{2}(\mathbf{I} + \frac{1}{2}F\bar{A}B^2 + F\bar{A}B|_k m^k) - L|\Omega|^2. \qquad (4.56)$$

Proof. By Proposition 4.12 vi) we have $E_{|\bar{s}}m^{\bar{s}} = -\frac{F\mathbf{K}}{2}\left(\bar{B} - F\bar{A}B\right)$, which substituted into vii) gives (4.56), taking into account (4.1). Their global validity complete the proof. □

Remark 4.4. *Under assumptions of above Corollary and* $B = 0$, *contracting (4.54) with* $\bar{m}^r m^j \bar{m}^h m^k m^l$ *we obtain* $\mathbf{K} = 0$. *On the other hand, by Proposition 4.1,* $\mathbf{I} = 0$. *Therefore, (4.56) becomes* $\mathbf{W} = -L\,|\Omega|^2$. *Further more* $\mathbf{W} = 0$ *if and only if* $\Omega = 0$.

Theorem 4.11. *Let* (M, F) *be a connected 2 - dimensional weakly Kähler complex Finsler space with* $R_{\bar{0}k\bar{0}0} = R_{\bar{0}0\bar{0}k}$, $|A| \neq 0$ *and*

$$\Phi_{|k} = \Phi[(J + Y)l_k + (V + H)m_k].$$

Then

$$K^h_{F,\lambda}(z, \eta) = 0 \text{ and } K^h_{F,\mu}(z, \eta) = -2L\,|\Omega|^2 \leq 0.$$

Moreover, $K^h_{F,\mu}(z, \eta)$ *is a constant if and only if* $\Omega = \frac{c}{F}$, *where* $c \in \mathbf{C}$.

Proof. It results by Corollary 4.4 and Proposition 4.12 vi) and vii). □

Theorem 4.12. *If* (M, F) *is a connected 2 - dimensional complex Berwald space with* $R_{\bar{0}k\bar{0}0} = R_{\bar{0}0\bar{0}k}$ *and* $|A| \neq 0$, *then* $R_{\bar{r}j\bar{h}k} = 0$.

Proof. The assumption of Berwald leads to $\Phi = \Omega = 0$. Applying now Proposition 4.12 and (4.46) we obtain $R_{\bar{r}j\bar{h}k} = 0$. □

Remark 4.5. *If* (M, F) *is a connected 2 - dimensional complex Berwald space with* $R_{\bar{0}k\bar{0}0} = R_{\bar{0}0\bar{0}k}$ *and* $|A| \neq 0$, *then the horizontal holomorphic sectional curvature in any direction is zero.*

Moreover, we obtain a known result from Hermitian geometry

Theorem 4.13. *If* (M, F) *is a 2 - dimensional Kähler purely Hermitian space with* $\mathbf{K}$ *a constant on* M, *then* $R_{\bar{r}j\bar{h}k} = \frac{\mathbf{K}}{2}(g_{j\bar{r}}g_{k\bar{h}} + g_{k\bar{r}}g_{j\bar{h}})$ *and the horizontal holomorphic sectional curvatures in directions* λ *and* μ *are* $2\mathbf{K}$.

Proof. Because $A = B = 0$, the Bianchi identity (4.54) is $R_{\bar{r}j\bar{h}k}|_l = 0$. Some contractions of it lead to

$-O_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}O(\bar{V} + \bar{H}) = H_{|\bar{0}} + \frac{F}{2}H(\bar{J} + \bar{Y}) = E_{|\bar{s}}m^{\bar{s}} = 0$,

$\frac{1}{F}E_{|\bar{0}} = V_{|\bar{s}}m^{\bar{s}} + \frac{1}{2}V(\bar{V} + \bar{H}) = \frac{1}{F}Y_{|\bar{0}} = -\frac{\mathbf{K}}{2}$ and $\mathbf{W} = \mathbf{K}$.

Therefore, (4.46) become

$R_{\bar{r}j\bar{h}k} = \frac{\mathbf{K}}{2}(2l_{\bar{r}}l_j l_{\bar{h}}l_k + 2m_{\bar{r}}m_j m_{\bar{h}}m_k + l_{\bar{r}}l_j m_{\bar{h}}m_k + m_{\bar{r}}l_j l_{\bar{h}}m_k + l_{\bar{r}}m_j m_{\bar{h}}l_k$
$+ m_{\bar{r}}m_j l_{\bar{h}}l_k) = \frac{\mathbf{K}}{2}(g_{j\bar{r}}g_{k\bar{h}} + g_{k\bar{r}}g_{j\bar{h}})$ and $K^h_{F,\lambda}(z, \eta) = K^h_{F,\mu}(z, \eta) = 2\mathbf{K}$. □

4.5.2 A particular case

Consider the problem of classifying the 2 - dimensional complex Finsler spaces for which

$$R_{\bar{r}j\bar{h}k} = \mathcal{K}(g_{j\bar{r}}g_{k\bar{h}} + g_{k\bar{r}}g_{j\bar{h}}), \tag{4.57}$$

where $\mathcal{K} = \mathcal{K}(z,\eta) : T'M \to \mathbf{R}$. We note that the 2 - dimensional complex Finsler spaces with (4.57) has the property that $K^h_{F,\lambda}(z,\eta) = K^h_{F,\mu}(z,\eta) = 4\mathcal{K}$. To solve the stated problem we use the Bianchi identities (4.54) and (4.52). Indeed, contracting the identity (4.54) by $\bar{\eta}^r\eta^j$ and taking into account

$R_{\bar{r}j\bar{h}k}|_l\bar{\eta}^r\eta^j = R_{\bar{0}0\bar{h}k}|_l - R_{\bar{0}l\bar{h}k}$
$= [L\mathcal{K}(2g_{k\bar{h}} - m_{\bar{h}}m_k)]\,|_l - F\mathcal{K}(2l_l l_{\bar{h}} l_k + l_l m_{\bar{h}} m_k + m_l m_{\bar{h}} l_k)$
$= F\mathcal{K}(2g_{k\bar{h}} - m_{\bar{h}}m_k)l_l + L\mathcal{K}|_l(2g_{k\bar{h}} - m_{\bar{h}}m_k)$
$+F\mathcal{K}l_k m_{\bar{h}} m_l - F\mathcal{K}(2l_l l_{\bar{h}} l_k + l_l m_{\bar{h}} m_k + m_l m_{\bar{h}} l_k)$
$= L\mathcal{K}|_l(2g_{k\bar{h}} - m_{\bar{h}}m_k)$;
$\Xi_{\bar{r}j\bar{h}l|k}\eta^j = P_{\bar{r}j\bar{s}k}P^{\bar{s}}_{\bar{0}l\bar{h}}\bar{\eta}^r = S_{\bar{r}j\bar{s}l}R^{\bar{s}}_{\bar{0}k\bar{h}}\bar{\eta}^r = 0$ and
$R_{\bar{r}j\bar{h}n}C^n_{kl}\bar{\eta}^r\eta^j = L\mathcal{K}(2g_{n\bar{h}} - m_{\bar{h}}m_n)(Al^n + Bm^n)m_k m_l$
$= L\mathcal{K}(2Al_{\bar{h}} + Bm_{\bar{h}})m_k m_l$, we obtain

$$\mathcal{K}|_l(2g_{k\bar{h}} - m_{\bar{h}}m_k) + \mathcal{K}(2Al_{\bar{h}} + Bm_{\bar{h}})m_k m_l = 0, \tag{4.58}$$

which is true in any local coordinates. Moreover, contraction by $\bar{\eta}^h\eta^k$ in (4.58) it follows $\mathcal{K}|_l = 0$, which means that $\mathcal{K} = \mathcal{K}(z)$ is a function of z alone. This implies that $\mathcal{K}(2Al_{\bar{h}} + Bm_{\bar{h}}) = 0$ and hence that either $A = 0$ (i.e. the purely Hermitian case) for any $\mathcal{K} = \mathcal{K}(z)$, or $|A| \neq 0$ and $\mathcal{K} = 0$ for any $|B|$.

So, we have proven

Corollary 4.6. *Let (M,F) be a connected 2 - dimensional complex Finsler space with $R_{\bar{r}j\bar{h}k} = \mathcal{K}(g_{j\bar{r}}g_{k\bar{h}} + g_{k\bar{r}}g_{j\bar{h}})$. Then it is purely Hermitian with $\mathcal{K}$ a function of z alone, or it is not purely Hermitian but with $\mathcal{K} = 0$.*

For $\mathcal{K} \neq 0$, (4.52) gives

$\mathcal{K}_{|l}(g_{j\bar{r}}g_{k\bar{h}} + g_{k\bar{r}}g_{j\bar{h}}) - \mathcal{K}_{|k}(g_{j\bar{r}}g_{l\bar{h}} + g_{l\bar{r}}g_{j\bar{h}}) + \mathcal{K}(g_{j\bar{r}}g_{n\bar{h}} + g_{n\bar{r}}g_{j\bar{h}})T^n_{kl} = 0$, which contracted by $\bar{\eta}^r\eta^j\bar{\eta}^h\eta^k$ leads to

$$F\mathcal{K}_{|l} - \mathcal{K}_{|0}l_l + F\mathcal{K}l^k l_n T^n_{kl} = 0. \tag{4.59}$$

Theorem 4.14. *Let (M,F) be a connected 2 - dimensional complex Finsler space with $R_{\bar{r}j\bar{h}k} = \mathcal{K}(g_{j\bar{r}}g_{k\bar{h}} + g_{k\bar{r}}g_{j\bar{h}})$, $\mathcal{K} \neq 0$. Then $\mathcal{K}$ is a constant on M if and only if (M,F) is Kähler.*

Proof. If $\mathcal{K}$ is a constant on (M, F) then, by (4.59) it results $l^k l_n T^n_{kl} = 0$ which leads to $T^n_{kl} = 0$, i.e. (M, F) is Kähler.

Conversely, if (M, F) is Kähler then by (4.59) we have $\mathcal{K}_{|l} = \frac{1}{F}\mathcal{K}_{|0} l_l$. Hence $\mathcal{K}_{|l} m^l = 0$. On the other hand, using the identity ii) of Proposition 2.2 it results $0 = \mathcal{K}|_{\bar{k}|j} = \mathcal{K}_{|j}|_{\bar{k}}$. Therefore,

$0 = \mathcal{K}_{|l}|_{\bar{r}} = -\frac{1}{2L}\mathcal{K}_{|0} l_l l_{\bar{r}} + \frac{1}{F}\mathcal{K}_{|0}|_{\bar{r}} l_l - \frac{1}{2L}\mathcal{K}_{|0} l_l l_{\bar{r}}$ and so

$-\frac{1}{L}\mathcal{K}_{|0} l_l l_{\bar{r}} = 0$ which gives $\mathcal{K}_{|0} = 0$. It follows $\mathcal{K}_{|l} = 0$ equivalently with $\frac{\partial \mathcal{K}}{\partial z^l} = 0$, i.e. $\mathcal{K}$ is a constant on (M, F). □

Now, we study the space with $|A| \neq 0$ and $\mathcal{K} = 0$ for any $|B|$. By (4.57) results $R_{\bar{r}j\bar{h}k} = 0$, and so the horizontal holomorphic sectional curvature in any direction vanishes identically. Moreover, the identity (4.54) become

$$\Xi_{\bar{r}j\bar{h}l|k} + P_{\bar{r}j\bar{s}k} P^{\bar{s}}_{\bar{0}l\bar{h}} = 0, \tag{4.60}$$

which contracted by $\bar{m}^r \eta^j \bar{\eta}^h \eta^k m^l$ and $\bar{m}^r \eta^j \bar{m}^h m^k m^l$ leads to $\Phi = \Omega = 0$, which are globally. Therefore, by Lemma 4.1 we obtain

Theorem 4.15. *If (M, F) is a connected 2 - dimensional complex Finsler space with $R_{\bar{r}j\bar{h}k} = 0$, $|A| \neq 0$, then $\dot{\partial}_{\bar{h}} G^i = 0$.*

Remark 4.6. *The converse of above Theorem is not true. There exists* 2 *- dimensional complex Finsler spaces with $\dot{\partial}_{\bar{h}} G^i = 0$ and $R_{\bar{r}j\bar{h}k} \neq 0$. A such example is given by the Antonelli - Shimada metric (4.33).*

Now, using Theorem 4.4 we have proven

Corollary 4.7. *If (M, F) is a connected 2 - dimensional weakly Kähler complex Finsler space with $R_{\bar{r}j\bar{h}k} = 0$, $|A| \neq 0$, then it is Berwald.*

Acknowledgment: This paper is supported by the Sectorial Operational Program Human Resources Development (SOP HRD), financed from the European Social Fund and by Romanian Government under the Project number POSDRU/89/1.5/S/59323.

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